Math 18.101
Supplementary Notes

Introduction

The change of variables formula in multi-variable calculus asserts that if U and V are open subsets of $\mathbb{R}^n$ and $f : U \to V$ a C^1 diffeomorphism then, for every continuous function, $\phi : V \to \mathbb{R}$ the integral

$$\int_V \phi(y)\, dy$$

exists if and only if the integral

$$\int_U \phi \circ f(x) |\det Df(x)|\, dx$$

exists, and if these integrals exist they are equal. A proof of this can be found in Chapter 4 of Munkres' book. These notes contain an alternative proof of this result. This proof is due to Peter Lax. Our version of his proof in the notes below makes use of the theory of differential forms; but, as Lax shows in the papers cited at the end of this section (which we strongly recommend as collateral reading for this course), references to differential forms can be avoided, and his proof can be couched entirely in the language of elementary multivariable calculus.

The virtue of Lax's proof is that it allows one to prove a version of the change of variables theorem for other mappings besides diffeomorphisms, and involves a topological invariant, the *degree of a mapping*, which is itself quite interesting. Some properties of this invariant, and some topological applications of the change of variables formula will be discussed in §6 of these notes.

Remark: The proof we are about to describe is somewhat simpler and more transparent if we assume that f is a C^2 diffeomorphism. We'll henceforth make this assumption.

Bibliography

1. P. Lax, Change of variables in multiple integrals I, Amer. Math. Monthly 106 (1999), 497-501.

2. ——, Change of variables in multiple integrals II, Amer. Math. Monthly 108 (2001), 115-119.

1 Sard's theorem

Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ a C^1 mapping. A point $p \in U$ is a *critical point* of f if the derivative of f at p,

$$Df(p) : \mathbb{R}^n \to \mathbb{R}^n,$$

is *not* bijective. Let C_f be the set of critical points of f. This set can be quite large. For instance if f is the constant mapping, $p \in U \to c$, then $Df(p) = 0$ for all $p \in U$; so $C_f = U$. Sard's theorem asserts, however, that no matter how big C_f is, its image with respect to f is "small". (For instance this is certainly the case in the example above where $f(C_f) = \{c\}$).

Theorem 1.1. *The image, $f(C_f)$, of the critical set of f is of measure zero in $\mathbb{R}^n$.*

Let $Q \subset U$ be a cube of width ℓ. To prove Sard's theorem we will first prove

Lemma 1.2. *Given $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(y) - f(x) - (Df)(x)(x - y)| \leq \epsilon |x - y| \tag{1.1}$$

for $x, y \in Q$ and $|x - y| \leq \delta$.

Proof. Df is a continuous function and Q is compact, so for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|Df(x) - Df(y)| \leq \frac{\epsilon}{n} \tag{1.2}$$

for $x, y \in Q$ and $|x - y| < \delta$. Moreover, by the mean value theorem, for every pair of points $x, y \in Q$ there exists a point z on the line joining x to y for which

$$f_i(y) - f_i(x) = Df_i(z)(y - x)\,. \tag{1.3}$$

Hence if $|x - y| < \delta$,

$$f_i(y) - f_i(x) - Df_i(x)(y - x) = (Df_i(z) - Df_i(x))(x - y)\,.$$

Thus

$$|f_i(y) - f_i(x) - Df_i(x)(y - x)| \leq n|Df(z) - Df(x)|\,|y - x| \leq \epsilon|y - x|\,.$$

$\square$

We will now prove that the set $f(C_f \cap Q)$ is of measure zero.

First of all note that, by (1.3),

$$|f(y) - f(x)| \leq c|x - y| \tag{1.4}$$

for $x, y \in Q$ where c is the supremum over Q of $n|Df|$. Now divide Q into N^n subcubes each of width, $\ell/N < \delta$. If R is one of these subcubes and R intersects C_f in a point x_0 then by (1.1),

$$|f(x) - f(x_0) - Df(x_0)(x - x_0)| \leq \epsilon |x - x_0| \tag{1.5}$$

for all $x \in R$. Let $y_0 = f(x_0)$ and let $A = Df(x_0)$. Then by (1.4) and (1.5), $f(R)$ is contained in the intersection of the set

$$|y - y_0 - A(y - y_0)| \leq \epsilon \tfrac{\ell}{N} \tag{1.6}$$

and the set

$$|y - y_0| < c \tfrac{\ell}{N}\,. \tag{1.7}$$

By assumption, A is not bijective; so its image is contained in a subspace W of $\mathbb{R}^n$ of dimension $(n-1)$. Let v be a unit vector perpendicular to W. If $y - y_0$ satisfies (1.6), its Euclidean distance to W is less than $\sqrt{n}\ell/N\epsilon$; i.e.,

$$|\langle y - y_0, v\rangle| \leq \sqrt{n} \tfrac{\ell}{N} \epsilon\,. \tag{1.8}$$

Let w be the orthogonal projection of $y - y_0$ onto W. Then

$$y - y_0 = av + w$$

with $a = \langle y - y_0, v\rangle$. Hence if $\epsilon < 1$, (1.7) and (1.8) imply

$$\|w\| \leq |a| + \|y - y_0\| \leq \sqrt{n} \tfrac{\ell}{N}(c + \epsilon) \leq 2\sqrt{n} \tfrac{\ell}{N} c\,.$$

This shows that the image of R is contained in a cylinder whose base B is an $(n-1)$-dimensional ball of radius $2\sqrt{n}\frac{\ell}{N}c$ and whose height, by (1.8) is $2\sqrt{n}\frac{\ell}{N}\epsilon$. Moreover, B in turn is contained in an $(n-1)$-dimensional cube whose sides are of length $4\sqrt{n}\frac{\ell}{N}c$; so we finally conclude: If $R \cap C_f$ is non-empty $f(R)$ is contained in a rectangular solid of volume

$$(4c)^{n-1}(\sqrt{n}\ell)^n \frac{2\epsilon}{N^n}\,.$$

Since there are at most N^n of these rectangles, $f(Q \cap C_f)$ is contained in a finite union of rectangular solids whose total volume is less than $2c_1\epsilon$, where

$$c_1 = (4c)^{n-1}(\sqrt{n}\ell)^n\,.$$

Since c and ℓ don't depend on ϵ, and since ϵ can be made arbitrarily small, it follows from this that $f(Q \cap C_f)$ is of measure zero. (See exercise 5.)

To conclude the proof of Sard's theorem let Q_i, $i = 1, 2, 3, \ldots$ be a covering of U by cubes. Then

$$f(C_f) = \bigcup_{i=1}^{\infty} f(Q_i \cap C_f)$$

and since each of the sets on the right is of measure zero, $f(C_f)$ is of measure zero.

Exercises for §1

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be the map: $f(x) = (x^2 - 1)^2$. What is the set of critical points of f? What is its image?

   (b) Same question for the map $f(x) = \sin x + x$.

   (c) Same question for the map

$$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases} .$$

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an affine map, i.e., a map of the form

$$f(x) = Ax + y_0$$

   where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map. Prove Sard's theorem for f.

3. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function which is supported in the interval $(-\frac{1}{2}, \frac{1}{2})$ and has a maximum at the origin. Let $r_1, r_2, r_3, \ldots$, be an enumeration of the rational numbers, and let $f : \mathbb{R} \to \mathbb{R}$ be the map

$$f(x) = \sum_{i=1}^{\infty} r_i \rho(x - i) .$$

   Show that f is a $\mathcal{C}^\infty$ map and show that the image of C_f is dense in $\mathbb{R}$. (The moral of this example: Sard's theorem says that the image of C_f is of measure zero, but the *closure* of this image can be quite large.)

4. Let S be a bounded subset of $\mathbb{R}^n$. Show that if S is rectifiable it can be covered by a finite number of rectangles of total volume, $\nu_S + \epsilon$, where ν_S is the volume of S.

   Hint: Let Q be a rectangle containing S in its interior. Recall that

$$\nu_S = \int_Q 1_S .$$

   Now let P be a partition of Q with $U(1_S, P) \leq \nu_S + \epsilon$.

5. Let S be a bounded subset of $\mathbb{R}^n$. Suppose that for every $\epsilon > 0$ there exists a rectifiable set, A, with $A \supset S$ and $\nu(A) < \epsilon$. Show that S is of measure zero.

6. Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^n$ be a C^1 mapping. Let q be an element of $\mathbb{R}^n$ *not* in $f(C_f)$. Show that if $p \in f^{-1}(q)$ there exists a neighborhood, U_0, of p with the property that p is the *only* point in U_0 that gets mapped onto q.

   Hint: The inverse function theorem.

7. Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^{n+k}$, $k \geq 1$, a C^1 mapping. Prove that $f(U)$ is of measure zero.

   Hint: Let $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be the projection, $\pi(x_1, \ldots, x_{n+k}) = (x_1, \ldots, x_n)$, and let $V = \pi^{-1}(U)$. Show that the mapping, $f \circ \pi : V \to \mathbb{R}^{n+k}$, has the same image as f and that $C_{f \circ \pi} = V$.

2 The Poincare lemma for rectangles

Let ν be a k-form on $\mathbb{R}^n$. We define the *support* of ν to be the closure of the set

$$\{x \in \mathbb{R}^n \,,\, \nu_x \neq 0\}$$

and we say that ν is *compactly* supported if supp ν is compact. Let

$$\omega = f\, dx_1 \wedge \ldots \wedge dx_n$$

be a compactly supported n-form with f continuous. We define the integral of ω over $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} \omega$$

to be the usual integral of f over $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} f\, dx\,.$$

(Since f is continuous and compactly supported this integral is well-defined.)

Now let Q be the rectangle

$$[a_1, b_1] \times \cdots \times [a_n, b_n]$$

and let $U = \operatorname{Int} Q$. The Poincare lemma for rectangles asserts:

Theorem 2.1. *Let ω be a compactly supported n-form of class C^r, $r \geq 1$, with* supp $\omega \subseteq U$. *Then the following assertions are equivalent:*

- *(a)* $\int \omega = 0$.
- *(b) There exists a compactly supported $(n-1)$-form, μ, of class C^r with* supp $\mu \subseteq U$ *satisfying $d\mu = \omega$.*

We will first prove that (b)⇒(a). Let

$$\mu = \sum_{i=1}^{n} f_i\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n\,,$$

(the "hat" over the dx_i meaning that dx_i has to be omitted from the wedge product). Then

$$d\mu = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n\,,$$

and to show that the integral of $d\mu$ is zero it suffices to show that each of the integrals

$$\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}\, dx \tag{2.1$_i$}$$

is zero. By Fubini we can compute $(2.1)_i$ by first integrating with respect to the variable x_i and then with respect to the remaining variables. But

$$\int \frac{\partial f}{\partial x_i}\, dx_i = f(x) \Big|_{x_i=a_i}^{x_i=b_i} = 0$$

since f_i is supported on U.

We will prove that (a) $\Rightarrow$ (b) by proving inductively the following assertion.

Lemma 2.2. *If ω is supported on U and its integral is zero there exists an $(n-1)$-form, μ, with* $\operatorname{supp}\,\mu \subseteq U$ *such that the form*

$$\omega - d\mu = f\, dx_1 \wedge \ldots \wedge dx_n \tag{2.2}$$

has the property

$$\int f(x_1, \ldots, x_{k-1}\,,\ x_k, \ldots, x_n)\, dx_k \cdots dx_n = 0\,. \tag*{$(2.3)_k$}$$

Note that for $k = n+1$ $(2.3)_k$ is just the assertion that $\omega - \ d\mu = 0$, i.e., $\omega = d\mu$, and for $k = 1$, $(2.3)_k$ is just the assertion (a). Let's prove that $(2.3)_k \Rightarrow (2.3)_{k+1}$.

Proof. Since $\omega - d\mu$ is supported on U its support is contained in a rectangle

$$[c_1, d_1] \times \cdots \times [c_n, d_n]$$

with $a_i < c_i < d_i < b_i$. Let

$$g(x_1, \ldots, x_k) = \int f(x_1, \ldots, x_k\,,\ x_{k+1}, \ldots, x_n)\, dx_{k+1} \ldots dx_n\,. \tag{2.4}$$

This function is compactly supported on the rectangle

$$Q' = [c_1, d_1] \times \cdots \times [c_k, d_k]$$

since the integrand on the right is zero when $(x_1, \ldots, x_k)$ lies outside Q'. Moreover, since f is of class C^r, this integral is of class C^r as a function of $x_1, \ldots, x_k$. (See exercise 3 below.) Let

$$h(x_1, \ldots, x_k) = \int_{a_k}^{x_k} g(x_1, \ldots, x_{k-1}, s)\, ds$$

for $a_k \leq x_k \leq b_k$. By $(2.3)_k$ and (2.4), $h(x_1, \ldots, x_k)$ is zero if $a_k \leq x_k \leq c_k$ or if $d_l \leq x_k \leq b_k$; so h is compactly supported on Q', is of class C^r and satisfies

$$\frac{\partial}{\partial x_k} h = g \tag{2.5}$$

by the fundamental theorem of calculus.

Now let $\rho = \rho(x_{k+1}, \dots, x_n)$ be a compactly supported function of class $\mathcal{C}^\infty$ with support on the rectangle

$$Q'' = [c_{k+1}, d_{k+1}] \times \cdots \times [c_n, d_n]$$

and with the property

$$\int_{Q''} \rho = 1 \,. \tag{2.6}$$

Let

$$\nu = (-1)^{k-1} h(x_1, \dots, x_k)\rho(x_{k+1}, \dots, x_n)\, dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n \,. \tag{2.7}$$

Then by (2.5) and (2.7)

$$d\nu = g(x_1, \dots, x_k)\rho(x_{k+1}, \dots, x_n)\, dx_1 \wedge \ldots \wedge dx_n$$

and hence by (2.2) and $(2.3)_k$

$$\omega - d(\mu + \nu) = f(x_1, \dots, x_n) - g(x_1, \dots, x_k)\rho(x_{k+1}, \dots, x_n)$$

and by (2.6) and (2.4)

$$\begin{aligned}\int f(x_1, \dots, x_n) - g(x_1, \dots, x_k)\rho(x_{k+1}, \dots, x_n)\, dx_{k+1} \dots dx_n& \\ = g(x_1, \dots, x_k) - g(x_1, \dots, x_k) = 0 \,,&\end{aligned}$$

so $\omega - d(\mu + \nu)$ satisfies $(2.3)_k$.

$\square$

Hence, by induction if ω satisfies the hypothesis (a) of Theorem 2.1 it satisfies $(2.3)_k$ for all k and in particular satisfies $(2.3)_{n+1}$ which just says that $\omega - d\mu = 0$.

Exercises for §2.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function of class C^r with support on the interval (a, b). Show that the following are equivalent.

(a) $\int_a^b f(x)\, dx = 0$.

(b) There exists a function, $g : \mathbb{R} \to \mathbb{R}$ of class C^{r+1} with support on (a, b) with $\frac{dg}{dx} = f$.

Hint: Show that the function

$$g(x) = \int_a^x f(s)\, ds$$

is compactly supported.

(2) Let $f = f(x,y)$ be a compactly supported function on $\mathbb{R}^k \times \mathbb{R}^\ell$ with the property that the partial derivatives

$$\frac{\partial f}{\partial x_i}(x,y)\,,\; i = 1,\ldots,k\,,$$

and are continuous as functions of x and y. Prove the following "differentiation under the integral sign" theorem.

Theorem 2.3. *The function*

$$g(x) = \int f(x,y)\,dy$$

is of class C^1 and

$$\frac{\partial g}{\partial x_i}(x) = \int \frac{\partial f}{\partial x_i}(x,y)\,dy\,.$$

Hints: For y fixed and $h \in \mathbb{R}^k$,

$$f_i(x+h,y) - f_i(x,y) = D_x f_i(c)h$$

for some point, c, on the line segment joining x to $x+c$. Using the fact that $D_x f$ is continuous as a function of x and y and compactly supported, conclude:

Lemma 2.4. *Given $\epsilon > 0$ there exists a $\delta > 0$ such that for $|h| \leq \delta$*

$$|f(x+h,y) - f(x,y) - D_x f(x,g)h| \leq \epsilon |h|\,.$$

Now let $Q \subseteq \mathbb{R}^\ell$ be a rectangle with $\operatorname{supp} f \subseteq \mathbb{R}^k \times Q$ and show that

$$|g(x+h) - g(x) - \left(\int D_x f(x,y)\,dy\right) h| \leq \epsilon \operatorname{vol}(Q)|h|\,.$$

Conclude that g is differentiable at x and that its derivative is

$$\int D_x f(x,y)\,dy\,.$$

(3) Let $f : \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}$ be a compactly supported continuous function. Prove

Theorem 2.5. *If all the partial derivatives of $f(x,y)$ with respect to x of order $\leq r$ exist and are continuous as functions of x and y the function*

$$g(x) = \int f(x,y)\,dy$$

is of class C^r.

3 The Poincare lemma for open subsets of $\mathbb{R}^n$

In this section we will generalize Theorem 2.1 to arbitrary connected open subsets of $\mathbb{R}^n$.

Theorem 3.1. *Let U be a connected open subset of $\mathbb{R}^n$ and let ω be a compactly supported n-form of class C^r with* supp $\omega \subset U$*. The the following assertions are equivalent,*

(a) $\int \omega = 0$.

(b) *There exists a compactly supported $(n-1)$-form μ of class C^r with* supp $\mu \subseteq U$ *and $\omega = d\mu$.*

Proof. Proof that (b) $\Rightarrow$ (a): Let $Q_i \subseteq U$, $i = 1, 2, 3, \ldots$, be a collection of rectangles with $U = \cup \operatorname{Int} Q_i$, and let ϕ_i, $i = 1, 2, 3, \ldots$ be a partition of unity with supp $\phi_i \subseteq \operatorname{Int} Q_i$. Since μ is compactly supported the sum, $\mu = \sum \phi_i \mu$, is finite; i.e., $\mu = \sum_{i=1}^{N} \phi_i \mu$ for N large enough. Hence

$$d\mu = \sum_{i=1}^{N} d(\phi_i \mu)$$

and since supp $\phi_i \mu \subseteq Q_i$

$$\int d\phi_i \mu = 0$$

by Theorem 2.1.

□

Proof that (a) $\Rightarrow$ (b): Let ω_1 and ω_2 be compactly supported n-forms of class C^r with support in U. We will write

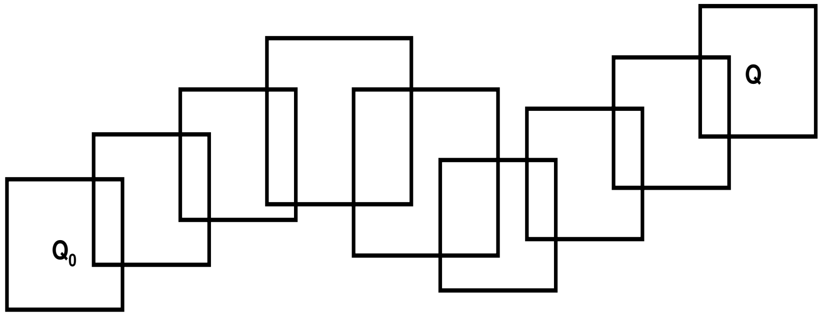
$$\omega_1 \sim \omega_2$$

as shorthand notation for the statement: "There exists a compactly supported $(n-1)$-form μ of class C^r with support in U and with $\omega_1 - \omega_2 = d\mu$.", We will prove that (a) $\Rightarrow$ (b) by proving an equivalent statement: Fix a rectangle $Q_0 \subset U$ and an n-form ω_0 of class C^r with supp $\omega_0 \subseteq Q_0$ and $\int \omega_0 = 1$.

Theorem 3.2. *If ω is a compactly supported n-form of class C^r with* supp $\omega \subseteq U$ *and $c = \int \omega$ then $\omega \sim c\omega_0$.*

Thus in particular if $c = 0$, Theorem 3.2 says that $\omega \sim 0$ proving that (a) $\Rightarrow$ (b).

To prove Theorem 3.2 let $Q_i \subseteq U$ be, as above, a collection of rectangles with $U = \cup \operatorname{Int} Q_i$ and let ϕ_i be a partition of unity with supp $\phi_i \subseteq \operatorname{Int} Q_i$. Replacing ω by the finite sum $\sum_{i=1}^{m} \phi_i \omega$, m large, it suffices to prove Theorem 3.2 for each of the summands $\phi_i \omega$. In other words we can assume that supp ω is contained in one of the open rectangles, Int Q_i. Denote this rectangle by Q. Using the fact that U is connected we will also show that we can join Q_0 to Q by a sequence of rectangles as in the figure below.



Lemma 3.3. *There exists a sequence of rectangles, R_i, $i = 0, \dots, N+1$ such that $R_0 = Q_0$, $R_{N+1} = Q$ and* Int $R_i \cap$ Int R_{i+1} *is non-empty.*

Proof. Denote by A the set of points, $x \in U$, for which there exists a sequence of rectangles, R_i, $i = 0, \dots, N+1$ with $R_0 = Q_0$, with $x \in$ Int R_{N+1} and with Int $R_i \cap$ Int R_{i+1} non-empty. It is clear that this set is open and that its complement is open; so, by the connectivity of U, $U = A$. $\square$

To prove Theorem 3.2 with supp $\omega \subseteq Q$, select, for each i, a compactly supported n-form ν_i of class C^r with supp $\nu_i \subseteq$ Int $R_i \cap$ Int R_{i+1} and with $\int \nu_i = 1$. The difference, $\nu_i - \nu_{i+1}$ is supported in Int R_{i+1}, and its integral is zero; so by Theorem 2.1, $\nu_i \sim \nu_{i+1}$. Similarly, $\omega_0 \sim \nu_1$ and, if $c = \int \omega$, $\omega \sim c\nu_N$. Thus,

$$c\omega_0 \sim c\nu_0 \sim \cdots \sim c\nu_N = \omega,$$

proving the theorem.

4 Proper mappings

Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$. A continuous mapping, $f : U \to V$, is *proper* if, for every compact subset B of V, $f^{-1}(B)$ is compact. Proper mappings have a number of nice properties which will be investigated in the exercises below. One obvious property is that if f is of class C^{r+1} and ω is a compactly supported k-form of class C^r with support on V, $f^*\omega$ is a compactly supported k-form of class C^r with support on U. Our goal in this section is to prove the following very general "change of variables theorem" for integrals of forms.

Theorem 4.1. *If U and V are connected open subsets of $\mathbb{R}^n$ and $f : U \to V$ is a proper C^2 mapping there exists a topological invariant of f called the* degree of f, *(denoted* $\deg(f)$*) with the property that, for every compactly supported n-form, ω, of class C^1 with support in V*

$$\int_U f^*\omega = \deg(f) \int_V \omega \,. \tag{4.1}$$

Before we prove this identity let's see what this "change of coordinates theorem" says in coordinates. If

$$\omega = \varphi(y)\, dy_1 \wedge \cdots \wedge dy_n$$

then at $x \in U$,

$$f^*\omega = (\varphi \circ f)(x) \det(Df(x))\, dx_1 \wedge \cdots \wedge dx_n \,;$$

so, in coordinates, (4.1) takes the form

$$\int_V \varphi(y)\, dy = \deg(f) \int_U \varphi \circ f(x) \det(Df(x))\, dx \,. \tag{4.2}$$

(See Munkres, §17.)

Proof. Theorem 4.1 is an easy consequence of the Poincaré lemma that we proved in §2. Let ω_0 be an n-form of class C^1 with compact support and with $\operatorname{supp} \omega_0 \subset V$ and with $\int \omega_0 = 1$. If we set $\deg f = \int_U f^*\omega_0$, then (4.1) clearly holds for ω_0. We will prove that (4.1) holds for every compactly supported n-form ω of class C^1 with $\operatorname{supp} \omega \subseteq V$. Let $c = \int_V \omega$. Then by Theorem 3.1 $\omega - c\omega_0 = d\mu$, where μ is an $(n-1)$-form of class C^1 with $\operatorname{supp} \mu \subseteq V$. Hence

$$f^*\omega - cf^*\omega_0 = f^*\, d\mu = df^*\mu \,,$$

and by part (a) of Theorem 3.1,

$$\int_U f^*\omega = c \int f^*\omega_0 = \deg(f) \int_V \omega \,.$$

$\square$

We will next discuss a multiplicative property of the degree which is useful for computational purposes. Let U, V and W be connected open subsets of $\mathbb{R}^n$ and $f : U \to V$ and $g : V \to W$ proper C^r mappings. We show below that

$$\deg(g \circ f) = \deg(g)\deg(f)\,. \tag{4.3}$$

Proof. Let ω be a compactly supported n-form of class C^1 with support on W. Then

$$(g \circ f)^*\omega = g^* f^* \omega\,;$$

so

$$\begin{aligned} \int_U (g \circ f)^*\omega &= \int_U g^*(f^*\omega) = \deg(g)\int_V f^*\omega \\ &= \deg(g)\deg(f)\int_W \omega\,. \end{aligned}$$

□

From this multiplicative property it is easy to deduce the following result (which we will need in the next section).

Theorem 4.2. *Let A be a non-singular $n \times n$ matrix and $f_A : \mathbb{R}^n \to \mathbb{R}^n$ the linear mapping associated with A. Then $\deg(f_A) = +1$ if $\det A$ is positive and -1 if $\det A$ is negative.*

A proof of this result is outlined in exercises 5–8 below.

Exercises for §4.

(1) Let U be an open subset of $\mathbb{R}^n$ and ϕ_i, $i = 1, 2, 3, \ldots$, a partition of unity on U. Show that the mapping, $f : U \to \mathbb{R}$ defined by

$$f = \sum_{k=1}^{\infty} k\phi_k$$

is a proper $\mathcal{C}^\infty$ mapping.

(2) Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$ and let $f : U \to V$ be a proper continuous mapping. Prove:

Theorem 4.3. *If B is a compact subset of V and $A = f^{-1}(B)$ then for every open subset, U_0, with $A \subseteq U_0 \subseteq U$, there exists an open subset, V_0, with $B \subseteq V_0 \subseteq V$ and $f^{-1}(V_0) \subseteq U_0$.*

Hint: Let C be a compact subset of V with $B \subseteq \text{Int } C$. Then the set $W = f^{-1}(C) - U_0$ is compact; so its image $f(W)$ is compact. Show that $f(W)$ and B are disjoint and let

$$V_0 = \text{Int } C - f(W)\,.$$

(3) Show that if $f : U \to V$ is a proper continuous mapping and X is a closed subset of U, then $f(X)$ is closed.

Hint: For $p \in V - f(X)$ let C be a compact subset of V with $p \in \text{Int } C$. Show that if $D = f^{-1}(C)$ then $f(X \cap D)$ is a compact set not containing p. Conclude that $\text{Int } C - f(X)$ is an open neighborhood of p in V.

(4) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the translation, $f(x) = x + a$. Show that $\deg(f) = 1$.

Hint: Let $\psi : \mathbb{R} \to \mathbb{R}$ be a compactly supported function of class C^1. The identity

$$\int \psi(t)\, dt = \int \psi(t-a)\, dt \tag{4.4}$$

is easy to prove by elementary calculus, and this identity proves the assertion above in dimension one. Now let

$$\phi(x) = \psi(x_1) \dots \psi(x_n) \tag{4.5}$$

and compute the right and left sides of (4.2) Fubini's theorem.

(5) Let σ be a permutation of the numbers, $1, \dots, n$ and let $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ be the diffeomorphism, $f_\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Prove that $\deg f_\sigma = \text{sgn}\,(\sigma)$.

Hint: Let ϕ be the function (4.5). Show that if $\omega = \phi(x)\, dx_1 \wedge \dots \wedge dx_n$, then $f^*\omega = (\text{sgn } \sigma)\omega$.

(6) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$f(x_1, \dots, x_n) = (x_1 + \lambda x_2, x_2, \dots, x_n).$$

Prove that $\deg(f) = 1$.

Hint: Let $\omega = \phi(x_1, \dots, x_n)\, dx_1 \wedge \dots \wedge dx_n$ where $\phi : \mathbb{R}^n \to \mathbb{R}$ is compactly supported and of class C^1. Show that

$$\int f^*\omega = \int \varphi(x_1 + x_2, x_2, \dots, x_n)\, dx_1 \dots dx_n$$

and evaluate the integral on the right by Fubini's theorem; i.e., by first integrating with respect to the x_1 variable and then with respect to the remaining variables. Note that by (4.4)

$$\int f(x_1 + \lambda x_2, x_2, \dots, x_n)\, dx_1 = \int f(x_1, x_2, \dots, x_n)\, dx_1\,.$$

(7) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$f(x_1, \dots, x_n) = (\lambda x_1, x_2, \dots, x_n)$$

with $\lambda \neq 0$. Show that $\deg f = +1$ if λ is positive and -1 if λ is negative.

Hint: In dimension 1 this is easy to prove by elementary calculus techniques. Prove it in d-dimensions by the same trick as in the previous exercise.

(8) Let A be a non-singular $n \times n$ matrix and $f_A : \mathbb{R}^n \to \mathbb{R}^n$ the linear mapping associated with A. Prove that $\deg(f_A) = +1$ if $\det A$ is positive and -1 if $\det A$ is negative.

Hint: Professor Munkres proves in §2 of chapter one that f_A can be written as a composition of linear mappings, $f_{E_1} \circ \cdots \circ f_{E_k}$ where the f_{E_k}'s are mappings of the type described in the previous three exercises. Now use the identity (4.3).

5 The change of variables formula

Let U and V be connected open subsets of $\mathbb{R}^n$. If $f : U \to V$ is a C^2-diffeomorphism, the determinant of $Df(x)$ at $x \in U$ is non-zero, and hence, since it is a continuous function of x, its sign is the same at every point. We will say that f is *orientation preserving* if this sign is positive and *orientation reversing* if it is negative. We will prove below:

Theorem 5.1. *The degree of f is $+1$ if f is orientation preserving and -1 if f is orientation reversing.*

We will then use this result to prove the following change of variables formula for diffeomorphisms.

Theorem 5.2. *Let $\phi : V \to \mathbb{R}$ be a compactly supported continuous function. Then*

$$\int_U \phi \circ f(x) |\det(Df)(x)| = \int_V \phi(y)\, dy \,. \tag{5.1}$$

Proof of Theorem 5.1. Given a point, $a_1 \in U$, let $a_2 = -f(a_1)$ and for $i = 1, 2$, let $g_i : \mathbb{R}^n \to \mathbb{R}^n$ be the translation, $g_i(x) = x + a_i$. By (4.1) and exercise 4 of § 4 the composite diffeomorphism

$$g_2 \circ f \circ g_1 \tag{5.2}$$

has the same degree as f, so it suffices to prove the theorem for this mapping. Notice however that this mapping maps the origin onto the origin. Hence, replacing f by this mapping, we can, without loss of generality, assume that 0 is in the domain of f and that $f(0) = 0$.

Next notice that if $A : \mathbb{R}^n \to \mathbb{R}^n$ is a bijective linear mapping the theorem is true for A (by exercise 8 of § 4), and hence if we can prove the theorem for $A^{-1} \circ f$, (4.1) will tell us that the theorem is true for f. In particular, letting $A = Df(0)$, we have

$$D(A^{-1} \circ f)(0) = A^{-1}Df(0) = I$$

where I is the identity mapping. Therefore, replacing f by $A^{-1}f$, we can assume that the mapping, f, for which we are attempting to prove Theorem 5.1 has the properties: $f(0) = 0$ and $Df(0) = I$. Let $g(x) = x - f(x)$. Then these properties imply that $g(0) = 0$ and $Df(0) = 0$.

□

Lemma 5.3. *There exists a $\delta > 0$ such that $|g(x)| \le \frac{1}{2}|x|$ for $|x| \le \delta$.*

Proof. Let $g(x) = (g_1(x), \dots, g_n(x))$. Then

$$\frac{\partial g_i}{\partial x_j}(0) = 0 \,;$$

so there exists a $\delta > 0$ such that

$$\left| \frac{\partial g_i}{\partial x_j}(x) \right| \leq \frac{1}{2\eta}$$

for $|x| \leq \delta$. However, by the mean value theorem,

$$g_i(x) = \sum \frac{\partial g_i}{\partial x_j}(c) x_j$$

for $c = t_0 x$, $0 < t_0 < 1$. Thus

$$|g_i(x)| \leq \frac{1}{2} \sup |x_i| = \frac{1}{2} |x| \, ,$$

so

$$|g(x)| = \sup |g_i(x)| \leq \frac{1}{2} |x| \, .$$

$\square$

Let ρ be a compactly supported $\mathcal{C}^\infty$ function with $0 \leq \rho \leq 1$ and with $\rho(x) = 0$ for $|x| \geq \delta$ and $\rho(x) = 1$ for $|x| \leq \frac{\delta}{2}$, and let $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping

$$\tilde{f}(x) = x + \rho(x) g(x) \, . \tag{5.3}$$

It's clear that

$$\tilde{f}(x) = x \text{ for } |x| \geq \delta \tag{5.4}$$

and, since $f(x) = x + g(x)$,

$$\tilde{f}(x) = f(x) \text{ for } |x| \leq \frac{\delta}{2} \, . \tag{5.5}$$

In addition, for all $x \in \mathbb{R}^n$:

$$|\tilde{f}(x)| \geq \frac{1}{2} |x| \, . \tag{5.6}$$

Indeed, by (5.4), $|\tilde{f}(x)| \geq |x|$ for $|x| \geq \delta$, and for $|x| \leq \delta$

$$\begin{aligned} |\tilde{f}(x)| &\geq |x| - \rho(x) |g(x)| \\ &\geq |x| - |g(x)| \geq |x| - \frac{1}{2} |x| = \frac{1}{2} |x| \end{aligned}$$

by Lemma 5.3.

Now let $\mathcal{Q}_r$ be the cube, $\{x \in \mathbb{R}^n \, , \, |x| \leq r\}$, and let $\mathcal{Q}_r^c = \mathbb{R}^n - \mathcal{Q}_r$.

From (5.6) we easily deduce that

$$\tilde{f}^{-1}(\mathcal{Q}_r) \subseteq \mathcal{Q}_{2r} \tag{5.7}$$

for all r, and hence that $\tilde{f}$ is *proper*. Also notice that for $x \in \mathcal{Q}_\delta$,

$$|\tilde{f}(x)| \leq |x| + |g(x)| \leq \frac{3}{2}\,|x|$$

by Lemma 5.3 and hence

$$\tilde{f}^{-1}(\mathcal{Q}^c_{\frac{3}{2}\delta}) \subseteq \mathcal{Q}^c_\delta \,. \tag{5.8}$$

Proof. We will now prove Theorem 5.1. Since f is a diffeomorphism mapping 0 to 0, it maps a neighborhood U_0 of 0 in U diffeomorphically onto a neighborhood V_0 of 0 in V, and by shrinking U_0 if necessary we can assume that U_0 is contained in $\mathcal{Q}_{\delta/2}$ and V_0 contained in $\mathcal{Q}_{\delta/4}$. Let ω be an n-form of class C^1 with support in V_0 whose integral over $\mathbb{R}^n$ is equal to one. Then $f^*\omega$ is supported in U_0 and hence in $\mathcal{Q}_{\delta/2}$. Also by (5.7) $\tilde{f}^*\omega$ is supported in $\mathcal{Q}_{\delta/2}$. Thus both of these forms are zero outside $\mathcal{Q}_{\delta/2}$. However, on $\mathcal{Q}_{\delta/2}$, $\tilde{f} = f$ by (5.5), so these forms are equal everywhere, and hence

$$\deg(f) = \int f^*\omega = \int \tilde{f}^*\omega = \deg(\tilde{f})\,.$$

Next let ω be a compactly supported n-form of class C^1 with support in $\mathcal{Q}^c_{3\delta/2}$ and with integral equal to one. Then $\tilde{f}^*\omega$ is supported in $\mathcal{Q}^c_\delta$ by (5.8), and hence since $f(x) = x$ on $\mathcal{Q}^c_\delta$ $\tilde{f}^*\omega = \omega$. Thus

$$\deg(\tilde{f}) = \int f^*\omega = \int \omega = 1\,.$$

Putting these two identities together we conclude that $\deg(f) = 1$. □

If the function ϕ in Theorem 5.2 is a C^1 function, the identity (5.1) is an immediate consequence of the result above and the identity (4.2). If ϕ is not C^1, but is just continuous, we will deduce Theorem 5.2 from the following result.

Theorem 5.4. *Let V be an open subset of $\mathbb{R}^n$. If $\phi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function of compact support with* supp $\phi \subseteq V$*; then for every $\epsilon > 0$ there exists a $\mathcal{C}^\infty$ function of compact support, $\psi : \mathbb{R}^n \to \mathbb{R}$ with* supp $\psi \subseteq V$ *and*

$$\sup |\psi(x) - \phi(x)| < \epsilon\,.$$

Proof. Let A be the support of ϕ and let d be the distance in the sup norm from A to the complement of V. Since ϕ is continuous and compactly supported it is uniformly continuous; so for every $\epsilon > 0$ there exists a $\delta > 0$ with $\delta < \frac{d}{2}$ such that $|\phi(x) - \phi(y)| < \epsilon$ when $|x - y| \leq \delta$. Now let Q be the cube: $|x| < \delta$ and let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a non-negative $\mathcal{C}^\infty$ function with supp $\rho \subseteq Q$ and

$$\int \rho(y)\,dy = 1\,. \tag{5.9}$$

Set

$$\psi(x) = \int \rho(y - x)\phi(y)\, dy\,.$$

By Theorem 2.5, ψ is a $\mathcal{C}^\infty$ function. Moreover, if A_δ is the set of points in $\mathbb{R}^d$ whose distance in the sup norm from A is $\leq \delta$ then for $x \notin A_\delta$ and $y \in A$, $|x - y| > \delta$ and hence $\rho(y - x) = 0$. Thus for $x \notin A_\delta$

$$\int \rho(y - x)\phi(y)\, dy = \int_A \rho(y - x)\phi(y)\, dy = 0\,,$$

so ψ is supported on the compact set A_δ. Moreover, since $\delta < \frac{d}{2}$, supp ψ is contained in V. Finally note that by (5.9) and exercise 4 of §4:

$$\int \rho(y - x)\, dy = \int \rho(y)\, dy = 1 \tag{5.10}$$

and hence

$$\phi(x) \;=\; \int \phi(x)\rho(y - x)\, dy$$

so

$$\phi(x) - \psi(x) \;=\; \int (\phi(x) - \phi(y))\rho(y - x)\, dy$$

and

$$|\phi(x) - \psi(x)| \;\leq\; \int |\phi(x) - \phi(y)|\, \rho(y - x)\, dy\,.$$

But $\rho(y-x) = 0$ for $|x-y| \geq \delta$; and $|\phi(x)-\phi(y)| < \epsilon$ for $|x-y| \leq \delta$, so the integrand on the right is less than

$$\epsilon \int \rho(y - x)\, dy\,,$$

and hence by (5.10)

$$|\phi(x) - \psi(x)| \leq \epsilon\,.$$

□

To prove the identity (5.1), let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$ cut-off function which is one on a neighborhood V_1 of the support of ϕ, is non-negative, and is compactly supported with supp $\gamma \subseteq V$, and let

$$c = \int \gamma(y)\, dy\,.$$

By Theorem 5.4 there exists, for every $\epsilon > 0$, a $\mathcal{C}^\infty$ function ψ, with support on V_1 satisfying

$$|\phi - \psi| \leq \tfrac{\epsilon}{2c}\,. \tag{5.11}$$

Thus

$$\begin{aligned}
\Big|\int_V (\phi-\psi)(y)\,dy\Big| &\le \int_V |\phi-\psi|(y)\,dy \\
&\le \int_V \gamma|\phi-\psi|(y)\,dy \\
&\le \tfrac{\epsilon}{2c}\int \gamma(y)\,dy \le \tfrac{\epsilon}{2}
\end{aligned}$$

so

$$\Big|\int_V \phi(y)\,dy - \int_V \psi(y)\,dy\Big| \le \tfrac{\epsilon}{2}\,. \tag{5.12}$$

Similarly, the expression

$$\Big|\int_U (\phi-\psi)\circ f(x)|\det Df(x)|\,dx\Big|$$

is less than or equal to the integral

$$\int_U \gamma\circ f(x)|(\phi-\psi)\circ f(x)|\;|\det Df(x)|\,dx$$

and by (5.11), $|(\phi-\psi)\circ f(x)| \le \frac{\epsilon}{2c}$, so this integral is less than or equal to

$$\tfrac{\epsilon}{2c}\int \gamma\circ f(x)|\det Df(x)|\,dx$$

and hence by (5.1) is less than or equal to $\frac{\epsilon}{2}$. Thus

$$\Big|\int_U \phi\circ f(x)\,|\det Df(x)|dx - \int_U \psi\circ f(x)|\det Df(x)|\;dx\Big| \le \tfrac{\epsilon}{2}\,. \tag{5.13}$$

Combining (5.12), (5.13) and the identity

$$\int_V \psi(y)\,dy = \int \psi\circ f(x)|\det Df(x)|\,dx$$

we get, for all $\epsilon > 0$,

$$\Big|\int_V \phi(y)\,dy - \int_U \phi\circ f(x)|\det Df(x)|\,dx\Big| \le \epsilon$$

and hence

$$\int \phi(y)\,dy = \int \phi\circ f(x)|\det Df(x)|\,dx\,.$$

Exercises for §5

(1) Let $h : V \to \mathbb{R}$ be a non-negative continuous function. Show that if the improper integral

$$\int_V h(y)\, dy$$

is well-defined, then the improper integral

$$\int_U h \circ f(x) |\det Df(x)|\, dx$$

is well-defined and these two integrals are equal.

Hint: If ϕ_i, $i = 1, 2, 3, \dots$ is a partition of unity on V then $\psi_i = \phi_i \circ f$ is a partition of unity on U and

$$\int \phi_i h\, dy = \int \psi_i (h \circ f(x)) |\det Df(x)|\, dx\,.$$

Now deduce the result above from Theorem 16.5 in Munkres' book.

(2) Show that the result above is true without the assumption that h is non-negative.

Hint: $h = h_+ - h_-$, where $h_+ = \max(h, 0)$ and $h_- = \max(-h, 0)$.

(3) Show that, in the formula (4.2), one can allow the function, ϕ, to be a *continuous* compactly supported function rather than a C^1 compactly supported function.

6 The degree of a differentiable mapping

Let U and V be open subsets of $\mathbb{R}^n$ and $f : U \to V$ a proper C^2 mapping. In this section we will show how to compute the degree of f and, in particular, show that it is always an integer. From this fact we will be able to conclude that the degree of f is a topological invariant of f: if we deform f smoothly, its degree doesn't change.

Definition 6.1. *Let C_f be the set of critical points of f. A point, $q \in V$, is a* regular value *of f if it is not in the image, $f(C_f)$, of C_f.*

By Sard's theorem "almost all" points, q in V are regular values of f; i.e., the set of points which are not regular values of f is a set of measure zero. Notice, by the way, that a point, q, can qualify as a regular value of f by *not* being in the image of f. For instance, for the constant map, $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(p) = c$, the points, $q \in \mathbb{R}^n - \{c\}$ are all regular values of f.

Picking a regular value, q, of f we will prove:

Theorem 6.2. *The set $f^{-1}(q)$ is a finite set. Moreover, if $f^{-1}(q) = \{p_1, \dots, p_n\}$, then there exist connected open neighborhoods U_i of p_i in Y and an open neighborhood W of q in V such that:*

(i) for $i \neq j$ U_i and U_j are disjoint;

(ii) $f^{-1}(W) = \bigcup U_i$,

(iii) f maps U_i diffeomorphically onto W.

Proof. If $p \in f^{-1}(q)$ then, since q is a regular value, $p \notin C_f$; so

$$Df(p) : \mathbb{R}^n \to \mathbb{R}^n$$

is bijective. Hence, by the inverse function theorem, f maps a neighborhood U_p of p diffeomorphically onto a neighborhood of q. The open sets

$$\{U_p\,, \quad p \in f^{-1}(q)\}$$

are a covering of $f^{-1}(q)$; and, since f is proper, $f^{-1}(q)$ is compact; so we can extract a finite subcovering

$$\{U_{p_i}\,, \quad i = 1, \dots, N\},$$

and since p_i is the only point in U_{p_i} which maps onto q, $f^{-1}(q) = \{p_1, \dots, p_N\}$.

Without loss of generality we can assume that the U_{p_i}'s are disjoint from each other; for, if not, we can replace them by smaller neighborhoods of the p_i's which have this property. By Theorem 4.3 there exists a connected open neighborhood W of q in V for which

$$f^{-1}(W) \subset \bigcup U_{p_i}\,.$$

To conclude the proof let $U_i = f^{-1}(W) \cap U_{p_i}$. $\square$

The main result of this section is a recipe for computing the degree of f by counting the number of p_i's above, keeping track of orientation.

Theorem 6.3. *For each $p_i \in f^{-1}(q)$ let $\sigma_{p_i} = +1$ if $f : U_i \to W$ is orientation preserving and -1 if $f : U_i \to W$ is orientation reversing. Then*

$$\deg(f) = \sum_{i=1}^{N} \sigma_{p_i} \,. \tag{6.1}$$

Proof. Let ω be a compactly supported n-form on W of class C^1 whose integral is one. Then

$$\deg(f) = \int_U f^*\omega = \sum_{i=1}^{N} \int_{U_i} f^*\omega \,.$$

Since $f : U_i \to W$ is a diffeomorphism

$$\int_{U_i} f^*\omega = \pm \int_W \omega = +1 \text{ or } -1$$

depending on whether $f : U_i \to W$ is orientation preserving or not. Thus $\deg(f)$ is equal to the sum (6.1).

□

As we pointed out above, a point, $q \in V$ can qualify as a regular value of f "by default", i.e., by not being in the image of f. In this case the recipe (6.1) for computing the degree gives "by default" the answer zero. Let's corroborate this directly.

Theorem 6.4. *If $f : U \to V$ isn't onto,* $\deg(f) = 0$.

Proof. By exercise 3 of §4, $V - f(U)$ is open; so if it is non-empty, there exists a compactly supported n-form, ω, of class $\mathcal{C}^\infty$ with support in $V - f(U)$ and with integral equal to one. Since $\omega = 0$ on the image of f, $f^*\omega = 0$; so

$$0 = \int_U f^*\omega = \deg(f) \int_V \omega = \deg(f) \,.$$

□

Remark: In applications the contrapositive of this theorem is much more useful than the theorem itself.

Theorem 6.5. *If* $\deg(f) \neq 0$, *then f maps U onto V.*

In other words if $\deg(f) \neq 0$ the equation

$$f(x) = y \tag{6.2}$$

has a solution, $x \in U$ for *every* $y \in V$.

We will now show that the degree of f is a topological invariant of f: if we deform f by a "homotopy" we don't change its degree. To prove this we will need a slight generalization of the notion of "proper mapping".

Definition 6.6. *Let X be a subset of $\mathbb{R}^m$ and Y a subset of $\mathbb{R}^m$. A continuous map $f : X \to Y$ is proper if for every compact subset, A, of Y, $f^{-1}(A)$ is compact.*

In particular, let U and V be open subsets of $\mathbb{R}^n$ and let a be a positive real number. Suppose that

$$g : [0, a] \times U \to V$$

is a proper C^1 mapping. For $t \in [0, a]$ let

$$f_t : U \to V \tag{6.3}$$

be the mapping, $f_t(p) = g(p, t)$. Then, if A is a compact subset of V, $f_t^{-1}(A)$ is the intersection of the compact set

$$\{(s, p) \in [0, a] \times U \, , \, g(s, p) \in A\}$$

with the set: $s = t$, and hence is compact. Therefore, for every $t \in [0, a]$, $f_t : U \to V$ is a proper C^1 mapping. If $f_0 = f$, we will call the family of mappings

$$f_t \, , \quad 0 \leq t \leq a$$

a “deformation” or “homotopy” of f.

Theorem 6.7. *For all $t \in [0, a]$, the degree of f_t is equal to the degree of f.*

Proof. Let

$$\omega = \phi(y) dy_1 \wedge \cdots \wedge dy_n$$

be a compactly supported n-form of class C^1 on U with integral equal to one. Then the degree of f_t is equal to the integral of $f_t^* \omega$ over U:

$$\int_U \phi(g_1(x, t), \ldots, g_n(x, t)) \det D_x g(x, t) \, dx \, . \tag{6.4}$$

The integrand in (6.4) is continuous and is supported on a compact subset of $[0, a] \times U$; hence (6.4) is continuous as a function of t. However, $\deg(f_t)$ is *integer* valued, so (6.4) is an (integer-valued) constant, not depending on t. $\square$

There are many other nice applications of Theorem 6.3. We'll content ourselves with two relatively simple and prosaic ones:

Application 1. The Brouwer fixed point theorem

Let B^n be the closed unit ball in $\mathbb{R}^n$:

$$\{x \in \mathbb{R}^n \, , \, \|x\| \leq 1\} \, .$$

Theorem 6.8. *If $f : B^n \to B^n$ is a C^2 mapping then f has a fixed point, i.e., f maps some point, $x_0 \in B^n$ onto itself.*

The idea of the proof will be to assume that there isn't a fixed point and show that this leads to a contradiction. Suppose that for every point, $x \in B^n$ $f(x) \neq x$. Consider the ray through $f(x)$ in the direction of x:

$$f(x) + s(x - f(x))\,, \quad 0 \leq s < \infty\,.$$

This intersects the boundary, S^{n-1}, of B^n in a unique point, $\gamma(x)$, (see figure 1 below); and one of the exercises at the end of this section will be to show that the mapping $\gamma : B^n \to S^{n-1}$, $x \to \gamma(x)$, is a C^2 mapping. Also it is clear from figure 1 that $\gamma(x) = x$ if $x \in S^{n-1}$.

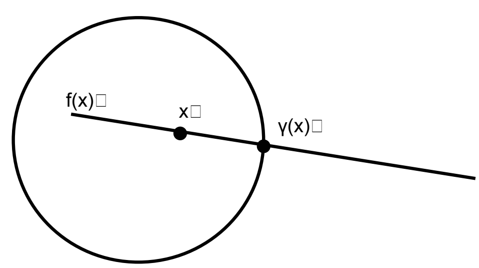


Figure 6.1.

Let $B^n(r)$ be the ball, $\{x \in \mathbb{R}^n\,,\ \|x\| \leq r\}$. Since γ is a C^2 mapping of B^n into $\mathbb{R}^n$, there exists an open set, U, containing B^n and a C^2 mapping of U into $\mathbb{R}^n$ whose restriction to B^n is γ. (For the sake of economy of notation we'll continue to call this map γ.) Since U is open and contains $B^n(1)$ it contains a slightly larger ball, $B^n(1+\delta_0)$, $\delta_0 > 0$. We claim:

Lemma 6.9. *Given $\epsilon > 0$ there exists $0 < \delta < \delta_0$ such that for $1 \leq \|x\| \leq 1+\delta$, $\|\gamma(x) - x\|$ is less than ϵ.*

Proof. Since γ is uniformly continuous on $B(1+\delta_0)$ there exists $0 < \delta < \delta_0$ such that for $x, y \in B(1+\delta_0)$ and $\|x-y\| < \delta$, $\|\gamma(x) - \gamma(y)\|$ is less than $\epsilon/2$. Moreover we can assume $\delta < \frac{\epsilon}{2}$. Let $1 \leq \|x\| \leq 1+\delta$ and let $y = x/\|x\|$. Then

$$x - y = \|x\|y - y = (\|x\| - 1)y$$

so

$$\|x-y\| = \|x\| - 1 \leq \delta\,.$$

Hence $\|\gamma(x)-\gamma(y)\|$ is less than $\epsilon/2$; therefore, since $\gamma(y)=y$, $\|\gamma(x)-y\|$ is less than $\epsilon/2$. Thus

$$\|\gamma(x)-x\| \leq \|\gamma(x)-y\| + \|y-x\| \leq \frac{\epsilon}{2} + \delta \leq \epsilon\,.$$

□

Now let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function, with $0 \leq \varphi \leq 1$, which is one on the set, $\|x\| \leq 1+\delta/2$, and zero on the set, $\|x\| \geq 1+\delta$, and let

$$g(x) = \varphi(x)\gamma(x) + (1-\varphi(x))x\,. \tag{6.5}$$

The mapping defined by (6.5) is a C^2 mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$ with the properties

$$g(x) = \gamma(x) \text{ for } \|x\| \leq 1+\delta/2 \tag{6.6}$$

and

$$g(x) = (x) \text{ for } \|x\| \geq 1+\delta\,. \tag{6.7}$$

We claim that on the set $1 \leq \|x\| \leq 1+\delta$

$$\|g(x)\| \geq 1-\epsilon\,. \tag{6.8}$$

Indeed, since $g(x) = \varphi(x)(\gamma(x)-x)+x$

$$\|g(x)\| \geq \|x\| - \|\gamma(x)-x\|$$

and by Lemma 6.9, $\|\gamma(x)-x\| < \epsilon$ if $1 \leq \|x\| \leq 1+\delta$. On the other hand, for $\|x\| \leq 1$, $g(x)=\gamma(x)$ and $\|\gamma(x)\|=1$ so $\|g(x)\|=1$ and for $\|x\| \geq 1+\delta$ $g(x)=x$, so $\|g(x)\| \geq 1+\delta$. Hence

$$\|g(x)\| \geq 1-\epsilon \tag{6.9}$$

for *all* x. Moreover, by (6.6), $g(x)$ is proper. Let's compute its degree. (6.8) tells us that if $\epsilon < 1$, the origin is *not* in the image of g, so by Theorem 6.4 the degree of g is zero. On the other hand g is equal to the identity map on the set $\|x\| \geq 1+\delta$, and the degree of the identity map is one, hence so is the degree of g. This gives us he contradiction we were looking for a proves by contradiction that f has to have a fixed point.

Application 2. The fundamental theorem of algebra

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients. If we identify the complex plane

$$\mathbb{C} = \{z = x+iy\,;\, x,y \in \mathbb{R}\}$$

with $\mathbb{R}^2$ via the map, $(x,y) \in \mathbb{R}^2 \to z = x+iy$, we can think of p as defining a mapping

$$p : \mathbb{R}^2 \to \mathbb{R}^2\,,\; z \to p(z)\,.$$

We will prove

Theorem 6.10. *The mapping, p, is proper and* $\deg(p) = n$.

Proof. For $0 \le t \le 1$ let

$$\begin{aligned} p_t(z) &= (1-t)z^n + tp(z) \\ &= z^n + t\sum_{i=0}^{n-1} a_i z^i \,. \end{aligned}$$

We will show that the mapping

$$g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2 \,,\ z \to p_t(z)$$

is a proper mapping. Let

$$C = \sup\{|a_i| \,,\ i = 0, \ldots, n-1\} \,.$$

Then for $|z| \ge 1$

$$\begin{aligned} |a_0 + \cdots + a_{n-1}z^{n-1}| &\le |a_0| + |a_1||z| + \cdots + |a_{n-1}|\,|z|^{n-1} \\ &\le C|z|^{n-1} \,, \end{aligned}$$

and hence, for $|z| > 2C$,

$$\begin{aligned} |p_t(z)| &\ge |z|^n - C|z|^{n-1} \\ &\ge C|z|^{n-1} \,. \end{aligned}$$

If A is a compact subset of $\mathbb{C}$ then for some $R > 0$, A is contained in the disk, $|w| \le R$ and hence $g^{-1}(A)$ is contained in the set

$$\{(t,z)\,;\ 0 \le t \le 1 \,,\ |p_t(z)| \le R\} \,;$$

and hence in the compact set

$$\{(t,z)\,;\ 0 \le t \le 1 \,,\ C|z|^{n-1} \le R\} \,;$$

and this shows that g is proper. Thus each of the mappings,

$$p_t : \mathbb{C} \to \mathbb{C} \,,$$

is proper and $\deg p_t = \deg p_1 = \deg p = \deg p_0$. However, $p_0 : \mathbb{C} \to \mathbb{C}$ is just the mapping, $z \to z^n$ and an elementary computation (see exercises 5 and 6 below) shows that the degree of this mapping is n.

□

In particular for $n > 0$ the degree of p is non-zero; so by Theorem 6.4 we conclude that $p : \mathbb{C} \to \mathbb{C}$ is surjective and hence has zero in its image.

Theorem 6.11. *(fundamental theorem of algebra)*
Every polynomial,

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \,,$$

with complex coefficients has a complex root.

Exercises for §6

(1) Let W be a subset of $\mathbb{R}^n$ and let $a(x)$, $b(x)$ and $c(x)$ be real-valued functions on W of class C^r. Suppose that for every $x \in W$ the quadratic polynomial

$$a(x)s^2 + b(x)s + c(x) \tag{*}$$

has two distinct real roots, $s_+(x)$ and $s_-(x)$, with $s_+(x) > s_-(x)$. Prove that s_+ and s_- are functions of class C^r.

Hint: What *are* the roots of the quadratic polynomial: $as^2 + bs + c$?

(2) Show that the function, $\gamma(x)$, defined in figure 1 is a C^1 mapping of B^n onto S^{2n-1}. *Hint:* $\gamma(x)$ lies on the ray,

$$f(x) + s(x - f(x)) , \quad 0 \le s < \infty,$$

and satisfies $\|\gamma(x)\| = 1$; so $\gamma(x)$ is equal to

$$f(x) + s_0(x - f(x)),$$

where s_0 is a non-negative root of the quadratic polynomial

$$\|f(x) + s(x - f(x))\|^2 - 1 .$$

Argue from figure 1 that this polynomial has to have two distinct real roots.

(3) Show that the Brouwer fixed point theorem isn't true if one replaces the closed unit ball by the open unit ball. *Hint:* Let U be the open unit ball (i.e., the interior of B^n). Show that the map

$$h : U \to \mathbb{R}^n , \quad h(x) = \frac{x}{1 - \|x\|^2}$$

is a diffeomorphism of U onto $\mathbb{R}^n$, and show that there are lots of mappings of $\mathbb{R}^n$ onto $\mathbb{R}^n$ which don't have fixed points.

(4) Show that the fixed point in the Brouwer theorem doesn't have to be an interior point of B^n, i.e., show that it can lie on the boundary.

(5) If we identify $\mathbb{C}$ with $\mathbb{R}^2$ via the mapping: $(x, y) \to z = x + iy$, we can think of a $\mathbb{C}$-linear mapping of $\mathbb{C}$ into itself, i.e., a mapping of the form

$$z \to cz , \quad c \in \mathbb{C}$$

as being an $\mathbb{R}$-linear mapping of $\mathbb{R}^2$ into itself. Show that the determinant of this mapping is $|c|^2$.

(6) (a) Let $f : \mathbb{C} \to \mathbb{C}$ be the mapping $f(z) = z^n$. Show that

$$Df(z) = nz^{n-1} .$$

Hint: Argue from first principles. Show that for $h \in \mathbb{C} = \mathbb{R}^2$

$$\frac{(z+h)^n - z^n - nz^{n-1}h}{|h|}$$

tends to zero as $|h| \to 0$.

(b) Conclude from the previous exercise that

$$\det Df(z) = n^2|z|^{2n-2} .$$

(c) Show that at every point $z \in \mathbb{C} - 0$, f is orientation preserving.

(d) Show that every point, $w \in \mathbb{C} - 0$ is a regular value of f and that

$$f^{-1}(w) = \{z_1, \ldots, z_n\},$$

with $\sigma_{z_i} = +1$.

(e) Conclude that the degree of f is n.