Lecture 20

We begin with a review of last lecture.

Consider a vector space V. A tensor $T \in \mathcal{L}^k$ is *decomposable* if $T = \ell_1 \otimes \cdots \otimes \ell_k$, $\ell_i \in \mathcal{L}^1 = V^*$. A decomposable tensor T is *redundant* of $\ell_i = \ell_{i+1}$ for some i. We define

$$\mathcal{I}^k = \mathcal{I}^k(V) = \text{Span}\{ \text{ redundant } k\text{-tensors } \}. \tag{4.94}$$

Because $\mathcal{I}^k \subseteq \mathcal{L}^k$, we can take the quotient space

$$\Lambda^k = \Lambda^k(V^*) = \mathcal{L}^k/\mathcal{I}^k, \tag{4.95}$$

defining the map

$$\pi : \mathcal{L}^k \to \Lambda^k. \tag{4.96}$$

We denote by $\mathcal{A}^k(V)$ the set of all alternating k-tensors. We repeat the main theorem from last lecture:

Theorem 4.27. *The map π maps $\mathcal{A}^k$ bijectively onto Λ^k. So, $\mathcal{A}^k \cong \Lambda^k$.*

It is easier to understand the space $\mathcal{A}^k$, but many theorems are much simpler when using Λ^k. This ends the review of last lecture.

4.6 Wedge Product

Now, let $T_1 \in \mathcal{I}^{k_1}$ and $T_2 \in \mathcal{L}^{k_2}$. Then $T_1 \otimes T_2$ and $T_2 \otimes T_1$ are in $\mathcal{I}^k$, where $k = k_1 + k_2$. The following is an example of the usefulness of Λ^k.

Let $\mu_i \in \Lambda^{k_i}$, $i = 1, 2$. So, $\mu_i = \pi(T_i)$ for some $T_i \in \mathcal{L}^{k_i}$. Define $k = k_1 + k_2$, so $T_1 \otimes T_2 \in \mathcal{L}^k$. Then, we define

$$\pi(T_1 \otimes T_2) = \mu_1 \wedge \mu_2 \in \Lambda^k. \tag{4.97}$$

Claim. *The product $\mu_i \wedge \mu_2$ is well-defined.*

Proof. Take any tensors $T_i' \in \mathcal{L}^{k_i}$ with $\pi(T_i') = \mu_i$. We check that

$$\pi(T_1' \otimes T_2') = \pi(T_1 \otimes T_2). \tag{4.98}$$

We can write

$$T_1' = T_1 + W_1, \text{ where } W_1 \in \mathcal{I}^{k_1}, \tag{4.99}$$

$$T_2' = T_2 + W_2, \text{ where } W_2 \in \mathcal{I}^{k_2}. \tag{4.100}$$

Then,

$$T_1' \otimes T_2' = T_1 \otimes T_2 + \underbrace{W_1 \otimes T_2 + T_1 \otimes W_2 + W_1 \otimes W_2}_{\in \mathcal{I}^k}, \tag{4.101}$$

so

$$\mu_1 \wedge \mu_2 \equiv \pi(T_1' \otimes T_2') = \pi(T_1 \otimes T_2). \tag{4.102}$$

□

This product ($\wedge$) is called the *wedge product*. We can define higher order wedge products. Given $\mu_i \in \Lambda^{k_i}$, $i = 1, 2, 3$, where $\mu = \pi(T_i)$, we define

$$\mu_1 \wedge \mu_2 \wedge \mu_3 = \pi(T_1 \otimes T_2 \otimes T_3). \tag{4.103}$$

We leave as an exercise to show the following claim.

Claim.

$$\begin{aligned} \mu_1 \wedge \mu_2 \wedge \mu_3 &= (\mu_1 \wedge \mu_2) \wedge \mu_3 \\ &= \mu_1 \wedge (\mu_2 \wedge \mu_3). \end{aligned} \tag{4.104}$$

Proof Hint: This triple product law also holds for the tensor product. □

We leave as an exercise to show that the two distributive laws hold:

Claim. *If $k_1 = k_2$, then*

$$(\mu_1 + \mu_2) \wedge \mu_3 = \mu_1 \wedge \mu_3 + \mu_2 \wedge \mu_3. \tag{4.105}$$

If $k_2 = k_3$, then

$$\mu_1 \wedge (\mu_2 + \mu_3) = \mu_1 \wedge \mu_2 + \mu_1 \wedge \mu_3. \tag{4.106}$$

Remember that $\mathcal{I}^1 = \{0\}$, so $\Lambda^1 = \Lambda^1/\mathcal{I}^1 = \mathcal{L}^1 = \mathcal{L}^1(V) = V^*$. That is, $\Lambda^1(V^*) = V^*$.

Definition 4.28. The element $\mu \in \Lambda^k$ is *decomposable* if it is of the form $\mu = \ell_1 \wedge \cdots \wedge \ell_k$, where each $\ell_i \in \Lambda^1 = V^*$.

That means that $\mu = \pi(\ell_1 \otimes \cdots \otimes \ell_k)$ is the projection of a decomposable k-tensor.

Take a permutation $\sigma \in S_k$ and an element $\omega \in \Lambda^k$ such that $\omega = \pi(T)$, where $T \in \mathcal{L}^k$.

Definition 4.29.

$$\omega^\sigma = \pi(T^\sigma). \tag{4.107}$$

We need to check that this definition does not depend on the choice of T.

Claim. *Define $\omega^\sigma = \pi(T^\sigma)$. Then,*

1. *The above definition does not depend on the choice of T,*

2. *$\omega^\sigma = (-1)^\sigma \omega$.*

Proof. 1. Last lecture we proved that for $T \in \mathcal{L}^k$,

$$T^\sigma = (-1)^\sigma T + W, \tag{4.108}$$

for some $W \in \mathcal{I}^k$. Hence, if $T \in \mathcal{I}^k$, then $T^\sigma \in \mathcal{I}^k$. If $\omega = \pi(T) = \pi(T')$, then $T' - T \in \mathcal{I}^k$. Thus, $(T')^\sigma - T^\sigma \in \mathcal{I}^k$, so $\omega^\sigma = \pi((T')^\sigma) = \pi(T^\sigma)$.

2.
$$T^\sigma = (-1)^\sigma T + W, \tag{4.109}$$

for some $W \in \mathcal{I}^k$, so
$$\pi(T^\sigma) = (-1)^\sigma \pi(T). \tag{4.110}$$

That is,
$$\omega^\sigma = (-1)^\sigma \omega. \tag{4.111}$$

□

Suppose ω is decomposable, so $\omega = \ell_1 \wedge \cdots \wedge \ell_k,\ \ell_i \in V^*$. Then $\omega = \pi(\ell_1 \wedge \cdots \wedge \ell_k)$, so

$$\begin{aligned} \omega^\sigma &= \pi((\ell_1 \otimes \cdots \otimes \ell_k)^\sigma) \\ &= \pi(\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}) \\ &= \ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)}. \end{aligned} \tag{4.112}$$

Using the previous claim,

$$\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k. \tag{4.113}$$

For example, if $k = 2$, then $\sigma = \tau_{1,2}$. So, $\ell_2 \wedge \ell_1 = -\ell_1 \wedge \ell_2$. In the case $k = 3$, we find that

$$\begin{aligned} (\ell_1 \wedge \ell_2) \wedge \ell_3 &= \ell_1 \wedge (\ell_2 \wedge \ell_3) \\ &= -\ell_1 \wedge (\ell_3 \wedge \ell_2) = -(\ell_1 \wedge \ell_3) \wedge \ell_2 \\ &= \ell_3 \wedge (\ell_1 \wedge \ell_2). \end{aligned} \tag{4.114}$$

This motivates the following claim, the proof of which we leave as an exercise.

Claim. *If $\mu \in \Lambda^2$ and $\ell \in \Lambda^1$, then*

$$\mu \wedge \ell = \ell \wedge \mu. \tag{4.115}$$

Proof Hint: Write out μ as a linear combination of decomposable elements of Λ^2. □

Now, suppose $k = 4$. Moving ℓ_3 and ℓ_4 the same distance, we find that

$$(\ell_1 \wedge \ell_2) \wedge (\ell_3 \wedge \ell_4) = (\ell_3 \wedge \ell_4) \wedge (\ell_1 \wedge \ell_2). \tag{4.116}$$

The proof of the following is an exercise.

Claim. *If $\mu \in \Lambda^2$ and $\nu \in \Lambda^2$, then*

$$\mu \wedge \nu = \nu \wedge \mu. \tag{4.117}$$

We generalize the above claims in the following:

Claim. *Left $\mu \in \Lambda^k$ and $\nu \in \Lambda^\ell$. Then*

$$\mu \wedge \nu = (-1)^{k\ell} \nu \wedge \mu. \tag{4.118}$$

Proof Hint: First assume k is even, and write out μ as a product of elements all of degree two. Second, assume that k is odd. □

Now we try to find a basis for $\Lambda^k(V^*)$. We begin with

$$e_1, \dots, e_n \text{ a basis of } V, \tag{4.119}$$

$$e_1^*, \dots, e_n^* \text{ a basis of } V^*, \tag{4.120}$$

$$e_I^* = e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*,\ I = (i_1, \dots, i_k), 1 \leq i_r \leq n,\ \text{ a basis of } \mathcal{L}^k, \tag{4.121}$$

$$\psi_I = \text{Alt}\,(e_I^*),\ \ I\text{'s strictly increasing},\ \text{ a basis of } \mathcal{A}^k(V). \tag{4.122}$$

We know that π maps $\mathcal{A}^k$ bijectively onto Λ^k, so $\pi(\psi_I)$, where I is strictly increasing, are a basis of $\Lambda^k(V^*)$.

$$\psi_I = \text{Alt } e_I^* = \sum (-1)^\sigma (e_I^*)^\sigma. \tag{4.123}$$

So,

$$\begin{aligned} \pi(\psi_I) &= \sum (-1)^\sigma \pi((e_I^*)^\sigma) \\ &= \sum (-1)^\sigma (-1)^\sigma \pi(e_I^*) \\ &= k! \pi(e_I^*) \\ &\equiv k! \tilde{e}_I. \end{aligned} \tag{4.124}$$

Theorem 4.30. *The elements of $\Lambda^k(V^*)$*

$$\tilde{e}_{i_1}^* \wedge \cdots \wedge \tilde{e}_{i_k}^*,\ 1 \leq i_1 < \ldots < i_k \leq n \tag{4.125}$$

are a basis of $\Lambda^k(V^*)$.

Proof. The proof is above. □

Let V, W be vector spaces, and let $A : V \to W$ be a linear map. We previously defined the pullback operator $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$. Also, given $T_i \in \mathcal{L}^{k_i}(W), i = 1, 2$, we showed that $A^*(T_1 \otimes T_2) = A^*T_1 \otimes A^*T_2$. So, if $T = \ell_1 \otimes \cdots \otimes \ell_k \in \mathcal{L}_k(W)$ is decomposable, then

$$A^*T = A^*\ell_1 \otimes \cdots \otimes A^*\ell_k,\ \ \ell_i \in W^*. \tag{4.126}$$

If $\ell_i = \ell_{i+1}$, then $A^*\ell_1 = A^*\ell_{i+1}$. This shows that if $\ell_1 \otimes \cdots \otimes \ell_k$ is redundant, then $A^*(\ell_1 \otimes \cdots \otimes \ell_k)$ is also redundant. So,

$$A^*\mathcal{I}^k(W) \subseteq \mathcal{I}^k(V). \tag{4.127}$$

Let $\mu \in \Lambda^k(W^*)$, so $\mu = \pi(T)$ for some $T \in \mathcal{L}^k(W)$. We can pullback to get $\pi(A^*T) \in \Lambda^k(V^*)$.

Definition 4.31. $A^*\mu = \pi(A^*T)$.

This definition makes sense. If $\mu = \pi(T) = \pi(T')$, then $T' - T \in \mathcal{I}^k(W)$. So $A^*T' - A^*T \in \mathcal{I}^k(V)$, which shows that $A^*\mu = \pi(A^*T') = \pi(A^*T)$.

We ask in the homework for you to show that the pullback operation is linear and that

$$A^*(\mu_1 \wedge \mu_2) = A^*\mu_1 \wedge A^*\mu_2. \tag{4.128}$$