Lecture 21

Let V, W be vector spaces, and let $A : V \to W$ be a linear map. We defined the pullback operation $A^* : W^* \to V^*$. Last time we defined another pullback operator having the form $A^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$. This new pullback operator has the following properties:

1. A^* is linear.
2. If $\omega_i \in \Lambda^{k_1}(W^*)$, $i = 1, 2$, then $A^*\omega_1 \wedge \omega_2 = A^*\omega_1 \wedge \omega_2$.
3. If ω is decomposable, that is if $\omega = \ell_1 \wedge \cdots \wedge \ell_k$ where $\ell_i \in W^*$, then $A^*\omega = A^*\ell_1 \wedge \cdots \wedge A^*\ell_k$.
4. Suppose that U is a vector space and that $B : W \to U$ is a linear map. Then, for every $\omega \in \Lambda^k(U^*)$, $A^*B^*\omega = (BA)^*\omega$.

4.7 Determinant

Today we focus on the pullback operation in the special case where $\dim V = n$. So, we are studying $\Lambda^n(V^*)$, which is called the *nth exterior power of V*.

Note that $\dim \Lambda^n(V^*) = 1$.

Given a linear map $A : V \to V$, what is the pullback operator

$$A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)? \tag{4.129}$$

Since it is a linear map from a one dimensional vector space to a one dimensional vector space, the pullback operator A^* is simply multiplication by some constant λ_A. That is, for all $\omega \in \Lambda^n(V^*)$, $A^*\omega = \lambda_A\omega$.

Definition 4.32. The *determinant of A* is

$$\det(A) = \lambda_A. \tag{4.130}$$

The determinant has the following properties:

1. If $A = I$ is the identity map, then $\det(A) = \det(I) = 1$.
2. If A, B are linear maps of V into V, then $\det(AB) = \det(A)\det(B)$.

   Proof: Let $\omega \in \Lambda^n(V^*)$. Then

$$\begin{aligned}(AB)^*\omega &= \det(AB)\omega \\ &= B^*(A^*\omega) \\ &= B^*(\det A)\omega \\ &= \det(A)\det(B)\omega.\end{aligned} \tag{4.131}$$

3. If A is onto, then $\det(A) \neq 0$.

   Proof: Suppose that $A : V \to V$ is onto. Then there exists an inverse linear map $A^{-1} : V \to V$ such that $AA^{-1} = I$. So, $\det(A)\det(A^{-1}) = 1$.

4. If A is not onto, then $\det(A) = 0$.

   Proof: Let $W = \text{Im}\,(A)$. If A is not onto, then $\dim W < \dim V$. Let $B : V \to W$ be the map A regarded as a map of V into W, and let $\iota_W : W \to V$ be inclusion. So, $A = \iota_W B$. For all $\omega \in \Lambda^n(V^*)$, $A^*\omega = B^*\iota_W^*\omega$. Note that $\iota_W^*\omega \in \Lambda^n(W^*) = \{0\}$ because $\dim W < n$. So, $A^*\omega = B^*\iota_W^* = 0$, which shows that $\det(A) = 0$.

Let W, V be n-dimensional vector spaces, and let $A : V \to W$ be a linear map. We have the bases

$$e_1, \dots, e_n \text{ basis of } V, \tag{4.132}$$
$$e_1^*, \dots, e_n^* \text{ dual basis of } V^*, \tag{4.133}$$
$$f_1, \dots, f_n \text{ basis of } W, \tag{4.134}$$
$$f_1^*, \dots, f_n^* \text{ dual basis of } W^*. \tag{4.135}$$

We can write $Ae_i = \sum a_{ij} f_j$, so that A has the associated matrix $A \sim [a_{ij}]$. Then $A^* f_j^* = \sum a_{jk} e_k^*$. Take $\omega = f_1^* \wedge \cdots \wedge f_n^* \in \Lambda^n(W^*)$, which is a basis vector of $\Lambda^n(W^*)$. Let us compute its pullback:

$$\begin{aligned} A^*(f_1^* \wedge \cdots \wedge f_n^*) &= \left(\sum_{k_1=1}^n a_{1,k_1} e_{k_1}^*\right) \wedge \cdots \wedge \left(\sum_{k_n=1}^n a_{n,k_n} e_{k_n}^*\right) \\ &= \sum_{k_1,\dots,k_n} (a_{1,k_1} \dots a_{m,k_n}) e_{k_1}^* \wedge \cdots \wedge e_{k_n}^*. \end{aligned} \tag{4.136}$$

Note that if $k_r = k_s$, where $r \neq s$, then $e_{k_1}^* \wedge \cdots \wedge e_{k_n}^* = 0$. If there are no repetitions, then there exists $\sigma \in S_n$ such that $k_i = \sigma(i)$. Thus,

$$\begin{aligned} A^*(f_1^* \wedge \cdots \wedge f_n^*) &= \sum_\sigma a_{1,\sigma(1)} \dots a_{n,\sigma(n)} e_{\sigma(1)}^* \wedge \cdots \wedge e_{\sigma(n)}^* \\ &= \left(\sum_\sigma (-1)^\sigma a_{1,\sigma(1)} \dots a_{n,\sigma(n)}\right) e_1^* \wedge \cdots \wedge e_n^*. \end{aligned} \tag{4.137}$$

Therefore,

$$\det[a_{ij}] = \sum_\sigma (-1)^\sigma a_{1,\sigma(1)} \dots a_{n,\sigma(n)}. \tag{4.138}$$

In the case where $W = V$ and each $e_i = f_i$, we set $\omega = e_1^* \wedge \cdots \wedge e_n^*$, and we get $A^*\omega = \det[a_{ij}]\omega$. So, $\det(A) = \det[a_{ij}]$.

For basic facts about determinants, see Munkres section 2. We will use these results quite a lot in future lectures. We list some of the basic results below.

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

1. $\det(A) = \det(A^t)$. You should prove this as an exercise. You should explain the following steps:

$$\begin{aligned}\det(A) &= \sum_\sigma (-1)^\sigma a_{1,\sigma(1)} \dots a_{n,\sigma(n)} \\ &= \sum_\tau (-1)^\tau a_{\tau(1),1} \dots a_{\tau(n),n}, \text{ where } \tau = \sigma^{-1} \\ &= \det(A^t).\end{aligned} \tag{4.139}$$

2. Let

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \tag{4.140}$$

where B is $k \times k$, C is $k \times \ell$, D is $\ell \times \ell$, and $n = k + \ell$. Then

$$\det(A) = \det(B)\det(D) \tag{4.141}$$

4.8 Orientations of Vector Spaces

Let $\ell \subseteq \mathbb{R}^2$ be a line through the origin. Then $\ell - \{0\}$ has two connected components. An *orientation* of ℓ is a choice of one of these components.

More generally, given a one-dimensional vector space $\mathbb{L}$, the set $\mathbb{L}$ has two connected components. Choose $v \in \mathbb{L} - \{0\}$. Then the two components are

$$\{\lambda v : \lambda \in \mathbb{R}_+\} \text{ and } \{-\lambda v : \lambda \in \mathbb{R}_+\}. \tag{4.142}$$

Definition 4.33. An *orientation of* $\mathbb{L}$ is a choice of one of these components, usually labeled $\mathbb{L}_+$. We define

$$v \in \mathbb{L}_+ \iff v \text{ is } \textit{positively oriented.} \tag{4.143}$$

Let V be an n-dimensional vector space. Then $\Lambda^n(V^*)$ is a 1-dimensional vector space.

Definition 4.34. An *orientation of* V is an orientation of $\Lambda^n(V^*)$. That is, a choice of $\Lambda^n(V^*)_+$.

Suppose $e_1, \dots, e_n$ is a basis of V, so $e_1^*, \dots, e_n^*$ is the dual basis of V^*. Let $\omega = e_1^* \wedge \dots \wedge e_n^* \in \Lambda^n(V^*) - \{0\}$.

Definition 4.35. The basis $e_1, \dots, e_n$ is *positively oriented* if $\omega \in \Lambda^n(V^*)_+$.

Let $f_1, \dots, f_n$ be another basis of V and $f_1^*, \dots, f_n^*$ its dual basis. Let $w' = f_1^* \wedge \dots \wedge f_n^*$. We ask: How is ω' related to ω? The answer: If $f_j = \sum a_{ij} e_i$, then $\omega' = \det[a_{ij}]\omega$. So, if $e_1, \dots, e_n$ is positively oriented, then $f_1, \dots, f_n$ is positively oriented if and only if $\det[a_{ij}] > 0$.

Suppose V is an n-dimensional vector space and that W is a k-dimensional subspace of V.

Claim. *If V and V/W are given orientations, then W acquires from these orientations a natural subspace orientation.*

Idea of proof: Let $\pi : V \to V/W$ be the canonical map, and choose a basis $e_1, \ldots, e_n$ of V such that $e_{\ell+1}, \ldots, e_n$ is a basis of W and such that $\pi(e_1), \ldots, \pi(e_\ell)$ is a basis of V/W, where $\ell = n - k$.

Replacing e_1 by $-e_1$ if necessary, we can assume that $\pi(e_1), \ldots, \pi(e_\ell)$ is an oriented basis of V/W. Replacing e_n by $-e_n$ if necessary, we can assume that $e_1, \ldots, e_n$ is an oriented basis of V. Now, give W the orientation for which $e_{\ell+1}, \ldots, e_n$ is an oriented basis of W. One should check that this choice of orientation for W is independent of the choice of basis (this is explained in the Multi-linear Algebra notes). $\square$