18.102 Introduction to Functional Analysis
Spring 2009

Problem 7.1 Question:- Is it possible to show the completeness of the Fourier basis

$$\exp(ikx)/\sqrt{2\pi}$$

by computation? Maybe, see what you think. These questions are also intended to get you to say things clearly.

(1) Work out the Fourier coefficients $c_k(t) = \int_{(0,2\pi)} f_t e^{-ikx}$ of the step function

$$f_t(x) = \begin{cases} 1 & 0 \le x < t \\ 0 & t \le x \le 2\pi \end{cases} \tag{16.17}$$

for each fixed $t \in (0, 2\pi)$.

(2) Explain why this Fourier series converges to f_t in $L^2(0, 2\pi)$ if and only if

$$2 \sum_{k>0} |c_k(t)|^2 = 2\pi t - t^2, \ t \in (0, 2\pi). \tag{16.18}$$

(3) Write this condition out as a Fourier series and apply the argument again to show that the completeness of the Fourier basis implies identities for the sum of k^{-2} and k^{-4}.

(4) Can you explain how reversing the argument, that knowledge of the sums of these two series should imply the completeness of the Fourier basis? There is a serious subtlety in this argument, and you get full marks for spotting it, without going ahead a using it to prove completeness.

Problem 7.2 Prove that for appropriate constants d_k, the functions $d_k \sin(kx/2)$, $k \in \mathbb{N}$, form an orthonormal basis for $L^2(0, 2\pi)$.

Hint: The usual method is to use the basic result from class plus translation and rescaling to show that $d'_k \exp(ikx/2)$ $k \in \mathbb{Z}$ form an orthonormal basis of $L^2(-2\pi, 2\pi)$. Then extend functions as odd from $(0, 2\pi)$ to $(-2\pi, 2\pi)$.

Problem 7.3 Let e_k, $k \in \mathbb{N}$, be an orthonormal basis in a separable Hilbert space, H. Show that there is a uniquely defined bounded linear operator $S : H \longrightarrow H$, satisfying

$$Se_j = e_{j+1} \ \forall \ j \in \mathbb{N}. \tag{16.19}$$

Show that if $B : H \longrightarrow H$ is a bounded linear operator then $S + \epsilon B$ is *not* invertible if $\epsilon < \epsilon_0$ for some $\epsilon_0 > 0$.

Hint:- Consider the linear functional $L : H \longrightarrow \mathbb{C}$, $Lu = (Bu, e_1)$. Show that $B'u = Bu - (Lu)e_1$ is a bounded linear operator from H to the Hilbert space $H_1 = \{u \in H; (u, e_1) = 0\}$. Conclude that $S + \epsilon B'$ is invertible as a linear map from H to H_1 for small ϵ. Use this to argue that $S + \epsilon B$ cannot be an isomorphism from H to H by showing that either e_1 is not in the range or else there is a non-trivial element in the null space.

Problem 7.4 Show that the product of bounded operators on a Hilbert space is strong continuous, in the sense that if A_n and B_n are strong convergent sequences of bounded operators on H with limits A and B then the product $A_n B_n$ is strongly convergent with limit AB.

Hint: Be careful! Use the result in class which was deduced from the Uniform Boundedness Theorem.