18.102 Introduction to Functional Analysis
Spring 2009

Lecture 8. Tuesday, Mar 3: Cauchy's inequality and Lebesgue measure

I first discussed the definition of preHilbert and Hilbert spaces and proved Cauchy's inequality and the parallelogram law. This can be found in all the lecture notes and many other places so I will not repeat it here. Another nice source is the book of G.F. Simmons, "Introduction to topology and modern analysis". I like it – but I think it is out of print.

In case anyone is interested in how to define Lebesgue measure from where we are now – and I may have time to do this later – we can just use the integral as I outlined on Tuesday. First, we define locally integrable functions. Thus $f : \mathbb{R} \longrightarrow \mathbb{C}$ is locally integrable if

$$F_{[-N,N]} = \begin{cases} f(x) & x \in [-N, N] \\ 0 & x \text{ if } |x| > N \end{cases} \in \mathcal{L}^1(\mathbb{R}) \ \forall \ N. \tag{8.1}$$

For example any continuous function on $\mathbb{R}$ is locally integrable.

Lemma 4. *The locally integrable functions form a linear space.*

Proof. Follows from the linearity of $\mathcal{L}^1(\mathbb{R})$. □

Definition 5. A set $A \subset \mathbb{R}$ is *measurable* if its characteristic function χ_A is locally integrable. A measurable set A has finite measure if $\chi_A \in \mathcal{L}^1(\mathbb{R})$ and then

$$\mu(A) = \int \chi_N \tag{8.2}$$

is the Lebesgue measure of A. If A is measurable but not of finite measure then $\mu(A) = \infty$ by definition.

We know immediately that any interval (a, b) is measurable (whether open, semi-open or closed) and has finite measure if and only if it is bounded – then the measure is $b - a$. Some things to check:-

Proposition 13. *The complement of a measurable set is measureable and any* countable *union of measurable sets is measurable.*

Proof. The first part follows from the fact that the constant function 1 is locally integrable and hence $\chi_{\mathbb{R}\setminus A} = 1 - \chi_A$ is locally integrable if and only if χ_A is locally integrable.

Notice the relationship between characteristic functions and the sets they define:-

$$\chi_{A\cup B} = \max(\chi_A, \chi_B), \ \chi_{A\cap B} = \min(\chi_A, \chi_B). \tag{8.3}$$

If we have a sequence of sets A_n then $B_n = \bigcup_{k\le n} A_k$ is clearly an increasing sequence of sets and

$$\chi_{B_n} \to \chi_B, \ B = \sum_n A_n \tag{8.4}$$

is an increasing sequence which converges pointwise (at each point it jumps to 1 somewhere and then stays or else stays at 0.) Now, if we multiply by $\chi_{[-N,N]}$ then

$$f_n = \chi_{[-N,N]}\chi_{B_n} \to \chi_{B\cap[-N,N]} \tag{8.5}$$

is an increasing sequence of integrable functions – assuming that is that the A_k's are measurable – with integral bounded above, by $2N$. Thus by our monotonicity theorem the limit is integrable so χ_B is locally integrable and hence $\bigcup_n A_n$ is measurable. □

Proposition 14. *The (Lebesgue) measurable subsets of* $\mathbb{R}$ *form a collection,* $\mathcal{M}$, *of the power set of* $\mathbb{R}$, *including* $\emptyset$ *and* $\mathbb{R}$ *which is closed under complements, countable unions and countable intersections.*

Proof. The countable intersection property follows from the others or directly by a similar argument to that above but with decreasing sequences. □

We have declared a set A which is measurable but not of finite measure to have infinite measure – for instance $\mathbb{R}$ is of infinite measure in this sense. Since the measure of a set is always non-negative (or undefined if it isn't measurable) this does not cause any problems and in fact Lebesgue measure is countable additive provided we allow ∞ as a value of the measure:-

$$A_n \in \mathcal{M},\ n \in \mathbb{N} \Longrightarrow \bigcup_n A_n \in \mathcal{M} \text{ and } mu(\bigcup_n A_n) \le \sum_n \mu(A_n) \tag{8.6}$$

with equality if $A_j \cap A_k = \emptyset$ for $j \neq k$.

It is a good exercise to prove this!

Problem set 4, Due 11AM Tuesday 10 Mar.

Just to compensate for last week, I will make this problem set too short and easy!

Problem 4.1

Let H be a normed space in which the norm satisfies the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2) \ \forall \ u, v \in H. \tag{8.7}$$

Show that the norm comes from a positive definite sesquilinear (i.e. Hermitian) inner product. Big Hint:- Try

$$(u,v) = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2\right)! \tag{8.8}$$

Problem 4.2

Let H be a finite dimensional (pre)Hilbert space. So, by definition H has a basis $\{v_i\}_{i=1}^n$, meaning that any element of H can be written

$$v = \sum_i c_i v_i \tag{8.9}$$

and there is no dependence relation between the v_i's – the presentation of $v = 0$ in the form (8.9) is unique. Show that H has an orthonormal basis, $\{e_i\}_{i=1}^n$ satisfying $(e_i, e_j) = \delta_{ij}$ ($= 1$ if $i = j$ and 0 otherwise). Check that for the orthonormal basis the coefficients in (8.9) are $c_i = (v, e_i)$ and that the map

$$T : H \ni v \longmapsto ((v, e_i)) \in \mathbb{C}^n \tag{8.10}$$

is a linear isomorphism with the properties

$$(u,v) = \sum_i (Tu)_i \overline{(Tv)_i}, \ \|u\|_H = \|Tu\|_{\mathbb{C}^n} \ \forall \ u, v \in H. \tag{8.11}$$

Why is a finite dimensional preHilbert space a Hilbert space?

Solutions to Problem set 3

This problem set is also intended to be a guide to what will be on the in-class test on March 5. In particular I will ask you to prove some of the properties of the Lebesgue integral, as below, plus one more abstract proof. Recall that equality a.e (almost everywhere) means equality on the complement of a set of measure zero.

Problem 3.1 If f and $g \in \mathcal{L}^1(\mathbb{R})$ are Lebesgue integrable functions on the line show that

(1) If $f(x) \geq 0$ a.e. then $\int f \geq 0$.
(2) If $f(x) \leq g(x)$ a.e. then $\int f \leq \int g$.
(3) If f is complex valued then its real part, $\operatorname{Re} f$, is Lebesgue integrable and $|\int \operatorname{Re} f| \leq \int |f|$.
(4) For a general complex-valued Lebesgue integrable function

$$|\int f| \leq \int |f|. \tag{8.12}$$

Hint: You can look up a proof of this easily enough, but the usual trick is to choose $\theta \in [0, 2\pi)$ so that $e^{i\theta} \int f = \int (e^{i\theta} f) \geq 0$. Then apply the preceeding estimate to $g = e^{i\theta} f$.

(5) Show that the integral is a continuous linear functional

$$\int : L^1(\mathbb{R}) \longrightarrow \mathbb{C}. \tag{8.13}$$

Solution:

(1) If f is real and f_n is a real-valued absolutely summable series of step functions converging to f where it is absolutely convergent (if we only have a complex-valued sequence use part (3)). Then we know that

$$g_1 = |f_1|,\ g_j = |f_j| - |f_{j-1}|,\ f \geq 1 \tag{8.14}$$

is an absolutely convergent sequence converging to $|f|$ almost everywhere. It follows that $f_+ = \frac{1}{2}(|f| + f) = f$, if $f \geq 0$, is the limit almost everywhere of the series obtained by interlacing $\frac{1}{2} g_j$ and $\frac{1}{2} f_j$:

$$h_n = \begin{cases} \frac{1}{2} g_k & n = 2k - 1 \\ f_k & n = 2k. \end{cases} \tag{8.15}$$

Thus f_+ is Lebesgue integrable. Moreover we know that

$$\int f_+ = \lim_{k\to\infty} \sum_{n\leq 2k} \int h_k = \lim_{k\to\infty} \int \left(|\sum_{j=1}^{k} f_j| + \sum_{j=1}^{k} f_j \right) \tag{8.16}$$

where each term is a non-negative step function, so $\int f_+ \geq 0$.

(2) Apply the preceeding result to $g - f$ which is integrable and satisfies

$$\int g - \int f = \int (g - f) \geq 0. \tag{8.17}$$

(3) Arguing from first principles again, if f_n is now complex valued and an absolutely summable series of step functions converging a.e. to f then define

$$h_n = \begin{cases} \operatorname{Re} f_k & n = 3k-2 \\ \operatorname{Im} f_k & n = 3k-1 \\ -\operatorname{Im} f_k & n = 3k. \end{cases} \tag{8.18}$$

This series of step functions is absolutely summable and

$$\sum_n |h_n(x)| < \infty \Longleftrightarrow \sum_n |f_n(x)| < \infty \Longrightarrow \sum_n h_n(x) = \operatorname{Re} f. \tag{8.19}$$

Thus $\operatorname{Re} f$ is integrable. Since $\pm \operatorname{Re} f \le |f|$

$$\pm \int \operatorname{Re} f \le \int |f| \Longrightarrow |\int \operatorname{Re} f| \le \int |f|. \tag{8.20}$$

(4) For a complex-valued f proceed as suggested. Choose $z \in \mathbb{C}$ with $|z| = 1$ such that $z\int f \in [0,\infty)$ which is possible by the properties of complex numbers. Then by the linearity of the integral

$$z\int f = \int (zf) = \int \operatorname{Re}(zf) \le \int |z \operatorname{Re} f| \le \int |f| \Longrightarrow |\int f| = z\int f \le \int |f|. \tag{8.21}$$

(where the second equality follows from the fact that the integral is equal to its real part).

(5) We know that the integral defines a linear map

$$I : L^1(\mathbb{R}) \ni [f] \longmapsto \int f \in \mathbb{C} \tag{8.22}$$

since $\int f = \int g$ if $f = g$ a.e. are two representatives of the same class in $L^1(\mathbb{R})$. To say this is continuous is equivalent to it being bounded, which follows from the preceeding estimate

$$|I([f])| = |\int f| \le \int |f| = \|[f]\|_{L^1} \tag{8.23}$$

(Note that writing $[f]$ instead of $f \in L^1(\mathbb{R})$ is correct but would normally be considered pedantic – at least after you are used to it!)

(6) I should have asked – and might do on the test: What is the norm of I as an element of the dual space of $L^1(\mathbb{R})$. It is 1 – better make sure that you can prove this.

Problem 3.2 If $I \subset \mathbb{R}$ is an interval, including possibly $(-\infty, a)$ or (a, ∞), we *define* Lebesgue integrability of a function $f : I \longrightarrow \mathbb{C}$ to mean the Lebesgue integrability of

$$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{C},\ \tilde{f}(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I. \end{cases} \tag{8.24}$$

The integral of f on I is then defined to be

$$\int_I f = \int \tilde{f}. \tag{8.25}$$

(1) Show that the space of such integrable functions on I is linear, denote it $\mathcal{L}^1(I)$.

(2) Show that is f is integrable on I then so is $|f|$.

(3) Show that if f is integrable on I and $\int_I |f| = 0$ then $f = 0$ a.e. in the sense that $f(x) = 0$ for all $x \in I \setminus E$ where $E \subset I$ is of measure zero as a subset of $\mathbb{R}$.
(4) Show that the set of null functions as in the preceeding question is a linear space, denote it $\mathcal{N}(I)$.
(5) Show that $\int_I |f|$ defines a norm on $L^1(I) = \mathcal{L}^1(I)/\mathcal{N}(I)$.
(6) Show that if $f \in \mathcal{L}^1(\mathbb{R})$ then

$$g : I \longrightarrow \mathbb{C},\ g(x) = \begin{cases} f(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I \end{cases} \tag{8.26}$$

is in $\mathcal{L}^1(\mathbb{R})$ an hence that f is integrable on I.
(7) Show that the preceeding construction gives a *surjective and continuous* linear map 'restriction to I'

$$L^1(\mathbb{R}) \longrightarrow L^1(I). \tag{8.27}$$

(Notice that these are the quotient spaces of integrable functions modulo equality a.e.)

Solution:

(1) If f and g are both integrable on I then setting $h = f + g$, $\tilde{h} = \tilde{f} + \tilde{g}$, directly from the definitions, so $f + g$ is integrable on I if f and g are by the linearity of $\mathcal{L}^1(\mathbb{R})$. Similarly if $h = cf$ then $\tilde{h} = c\tilde{f}$ is integrable for any constant c if $\tilde{f}$ is integrable. Thus $\mathcal{L}^1(I)$ is linear.
(2) Again from the definition, $|\tilde{f}| = \tilde{h}$ if $h = |f|$. Thus f integrable on I implies $\tilde{f} \in \mathcal{L}^1(\mathbb{R})$, which, as we know, implies that $|\tilde{f}| \in \mathcal{L}^1(\mathbb{R})$. So in turn $\tilde{h} \in \mathcal{L}^1(\mathbb{R})$ where $h = |f|$, so $|f| \in \mathcal{L}^1(I)$.
(3) If $f \in \mathcal{L}^1(I)$ and $\int_I |f| = 0$ then $\int_{\mathbb{R}} |\tilde{f}| = 0$ which implies that $\tilde{f} = 0$ on $\mathbb{R} \setminus E$ where $E \subset \mathbb{R}$ is of measure zero. Now, $E_I = E \cap I \subset E$ is also of measure zero (as a subset of a set of measure zero) and f vanishes outside E_I.
(4) If $f, g : I \longrightarrow \mathbb{C}$ are both of measure zero in this sense then $f + g$ vanishes on $I \setminus (E_f \cup E_g)$ where $E_f \subset I$ and $E_f \subset I$ are of measure zero. The union of two sets of measure zero (in $\mathbb{R}$) is of measure zero so this shows $f + g$ is null. The same is true of $cf + dg$ for constant c and d, so $\mathcal{N}(I)$ is a linear space.
(5) If $f \in \mathcal{L}^1(I)$ and $g \in \mathcal{N}(I)$ then $|f+g| - |f| \in \mathcal{N}(I)$, since it vanishes where g vanishes. Thus

$$\int_I |f + g| = \int_I |f| \ \forall\ f \in \mathcal{L}^1(I),\ g \in \mathcal{N}(I). \tag{8.28}$$

Thus

$$\|[f]\|_I = \int_I |f| \tag{8.29}$$

is a well-defined function on $L^1(I) = \mathcal{L}^1(\mathbb{R})/\mathcal{N}(I)$ since it is constant on equivalence classes. Now, the norm properties follow from the same properties on the whole of $\mathbb{R}$.

(6) Suppose $f \in \mathcal{L}^1(\mathbb{R})$ and g is defined in (8.26) above by restriction to I. We need to show that $g \in \mathcal{L}^1(\mathbb{R})$. If f_n is an absolutely summable series of step functions converging to f wherever, on $\mathbb{R}$, it converges absolutely consider

$$g_n(x) = \begin{cases} f_n(x) & \text{on } \tilde{I} \\ 0 & \text{on } \mathbb{R} \setminus \tilde{I} \end{cases} \tag{8.30}$$

where $\tilde{I}$ is I made half-open if it isn't already – by adding the lower end-point (if there is one) and removing the upper end-point (if there is one). Then g_n is a step function (which is why we need $\tilde{I}$). Moreover, $\int |g_n| \le \int |f_n|$ so the series g_n is absolutely summable and converges to g_n outside I and at all points inside I where the series is absolutely convergent (since it is then the same as f_n). Thus g is integrable, and since $\tilde{f}$ differs from g by its values at two points, at most, it too is integrable so f is integrable on I by definition.

(7) First we check we do have a map. Namely if $f \in \mathcal{N}(\mathbb{R})$ then g in (8.26) is certainly an element of $\mathcal{N}(I)$. We have already seen that 'restriction to I' maps $\mathcal{L}^1(\mathbb{R})$ into $\mathcal{L}^1(I)$ and since this is clearly a linear map it defines (8.27) – the image only depends on the equivalence class of f. It is clearly linear and to see that it is surjective observe that if $g \in \mathcal{L}^1(I)$ then extending it as zero outside I gives an element of $\mathcal{L}^1(\mathbb{R})$ and the class of this function maps to $[g]$ under (8.27).

Problem 3.3 Really continuing the previous one.

(1) Show that if $I = [a, b)$ and $f \in L^1(I)$ then the restriction of f to $I_x = [x, b)$ is an element of $L^1(I_x)$ for all $a \le x < b$.
(2) Show that the function

$$F(x) = \int_{I_x} f : [a, b) \longrightarrow \mathbb{C} \tag{8.31}$$

is continuous.
(3) Prove that the function $x^{-1}\cos(1/x)$ is not Lebesgue integrable on the interval $(0, 1]$. Hint: Think about it a bit and use what you have shown above.

Solution:

(1) This follows from the previous question. If $f \in L^1([a, b))$ with f' a representative then extending f' as zero outside the interval gives an element of $\mathcal{L}^1(\mathbb{R})$, by defintion. As an element of $L^1(\mathbb{R})$ this does not depend on the choice of f' and then (8.27) gives the restriction to $[x, b)$ as an element of $L^1([x, b))$. This is a linear map.
(2) Using the discussion in the preceeding question, we now that if f_n is an absolutely summable series converging to f' (a representative of f) where it converges absolutely, then for any $a \le x \le b$, we can define

$$f'_n = \chi([a, x))f_n,\ f''_n = \chi([x, b))f_n \tag{8.32}$$

where $\chi([a,b))$ is the characteristic function of the interval. It follows that f'_n converges to $f\chi([a,x))$ and f''_n to $f\chi([x,b))$ where they converge absolutely. Thus

$$\int_{[x,b)} f = \int f\chi([x,b)) = \sum_n \int f''_n,\ \int_{[a,x)} f = \int f\chi([a,x)) = \sum_n \int f'_n. \tag{8.33}$$

Now, for step functions, we know that $\int f_n = \int f'_n + \int f''_n$ so

$$\int_{[a,b)} f = \int_{[a,x)} f + \int_{[x,b)} f \tag{8.34}$$

as we have every right to expect. Thus it suffices to show (by moving the end point from a to a general point) that

$$\lim_{x\to a} \int_{[a,x)} f = 0 \tag{8.35}$$

for any f integrable on $[a,b)$. Thus can be seen in terms of a defining absolutely summable sequence of step functions using the usual estimate that

$$|\int_{[a,x)} f| \le \int_{[a,x)} |\sum_{n\le N} f_n| + \sum_{n>N} \int_{[a,x)} |f_n|. \tag{8.36}$$

The last sum can be made small, independent of x, by choosing N large enough. On the other hand as $x \to a$ the first integral, for fixed N, tends to zero by the definition for step functions. This proves (8.36) and hence the continuity of F.

(3) If the function $x^{-1}\cos(1/x)$ were Lebesgue integrable on the interval $(0,1]$ (on which it is defined) then it would be integrable on $[0,1)$ if we define it arbitrarily, say to be 0, at 0. The same would be true of the absolute value and Riemann integration shows us easily that

$$\lim_{t\downarrow 0} \int_t^1 x|\cos(1/x)|dx = \infty. \tag{8.37}$$

This is contrary to the continuity of the integral as a function of the limits just shown.

Problem 3.4 [Harder but still doable] Suppose $f \in \mathcal{L}^1(\mathbb{R})$.

(1) Show that for each $t \in \mathbb{R}$ the translates

$$f_t(x) = f(x-t) : \mathbb{R} \longrightarrow \mathbb{C} \tag{8.38}$$

are elements of $\mathcal{L}^1(\mathbb{R})$.

(2) Show that

$$\lim_{t\to 0} \int |f_t - f| = 0. \tag{8.39}$$

This is called 'Continuity in the mean for integrable functions'. Hint: I will add one!

(3) Conclude that for each $f \in \mathcal{L}^1(\mathbb{R})$ the map (it is a 'curve')

$$\mathbb{R} \ni t \longmapsto [f_t] \in L^1(\mathbb{R}) \tag{8.40}$$

is continuous.

Solution:

(1) If f_n is an absolutely summable series of step functions converging to f where it converges absolutely then $f_n(\cdot - t)$ is such a series converging to $f(\cdot - t)$ for each $t \in \mathbb{R}$. Thus, each of the $f(x-t)$ is Lebesgue integrable, i.e. are elements of $\mathcal{L}^1(\mathbb{R})$

(2) Now, we know that if f_n is a series converging to f as above then

$$\int |f| \le \sum_n \int |f_n|. \tag{8.41}$$

We can sum the first terms and then start the series again and so it follows that for any N,

$$\int |f| \le \int |\sum_{n\le N} f_n| + \sum_{n>N} \int |f_n|. \tag{8.42}$$

Applying this to the series $f_n(\cdot - t) - f_n(\cdot)$ we find that

$$\int |f_t - f| \le \int |\sum_{n\le N} f_n(\cdot - t) - f_n(\cdot)| + \sum_{n>N} \int |f_n(\cdot - t) - f_n(\cdot)| \tag{8.43}$$

The second sum here is bounded by $2 \sum\limits_{n>N} \int |f_n|$. Given $\delta > 0$ we can choose N so large that this sum is bounded by $\delta/2$, by the absolute convergence. So the result is reduce to proving that if $|t|$ is small enough then

$$\int |\sum_{n\le N} f_n(\cdot - t) - f_n(\cdot)| \le \delta/2. \tag{8.44}$$

This however is a finite sum of step functions. So it suffices to show that

$$|\int g(\cdot - t) - g(\cdot)| \to 0 \text{ as } t \to 0 \tag{8.45}$$

for each component, i.e. a constant, c, times the characteristic function of an interval $[a, b)$ where it is bounded by $2|c||t|$.

(3) For the 'curve' f_t which is a map

$$\mathbb{R} \ni t \longmapsto f_t \in \mathcal{L}^1(\mathbb{R}) \tag{8.46}$$

it follows that $f_{t+s} = (f_t)_s$ so we can apply the argument above to show that for each s,

$$\lim_{t\to s} \int |f_t - f_s| = 0 \Longrightarrow \lim_{t\to s} \|[f_t] - [f_s]\|_{L^1} = 0 \tag{8.47}$$

which proves continuity of the map (8.46).

Problem 3.5 In the last problem set you showed that a continuous function on a compact interval, extended to be zero outside, is Lebesgue integrable. Using this, and the fact that step functions are dense in $L^1(\mathbb{R})$ show that the linear space of continuous functions on $\mathbb{R}$ each of which vanishes outside a compact set (which depends on the function) form a dense subset of $L^1(\mathbb{R})$.

Solution: Since we know that step functions (really of course the equivalence classes of step functions) are dense in $L^1(\mathbb{R})$ we only need to show that any step function is the limit of a sequence of continuous functions each vanishing outside a compact set, with respect to L^1. So, it suffices to prove this for the charactertistic function of an interval $[a,b)$ and then multiply by constants and add. The sequence

$$g_n(x) = \begin{cases} 0 & x < a - 1/n \\ n(x - a + 1/n) & a - 1/n \leq x \leq a \\ 0 & a < x < b \\ n(b + 1/n - x) & b \leq x \leq b + 1/n \\ 0 & x > b + 1/n \end{cases} \tag{8.48}$$

is clearly continuous and vanishes outside a compact set. Since

$$\int |g_n - \chi([a,b))| = \int_{a-1/n}^{1} g_n + \int_{b}^{b+1/n} g_n \leq 2/n \tag{8.49}$$

it follows that $[g_n] \to [\chi([a,b))]$ in $L^1(\mathbb{R})$. This proves the density of continuous functions with compact support in $L^1(\mathbb{R})$.

Problem 3.6

(1) If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is bounded and continuous and $f \in \mathcal{L}^1(\mathbb{R})$ show that $gf \in \mathcal{L}^1(\mathbb{R})$ and that

$$\int |gf| \leq \sup_{\mathbb{R}} |g| \cdot \int |f|. \tag{8.50}$$

(2) Suppose now that $G \in \mathcal{C}([0,1] \times [0,1])$ is a continuous function (I use $\mathcal{C}(K)$ to denote the continuous functions on a compact metric space). Recall from the preceeding discussion that we have defined $L^1([0,1])$. Now, using the first part show that if $f \in L^1([0,1])$ then

$$F(x) = \int_{[0,1]} G(x, \cdot) f(\cdot) \in \mathbb{C} \tag{8.51}$$

(where $\cdot$ is the variable in which the integral is taken) is well-defined for each $x \in [0,1]$.

(3) Show that for each $f \in L^1([0,1])$, F is a continuous function on $[0,1]$.

(4) Show that

$$L^1([0,1]) \ni f \longmapsto F \in \mathcal{C}([0,1]) \tag{8.52}$$

is a bounded (i.e. continuous) linear map into the Banach space of continuous functions, with supremum norm, on $[0,1]$.

Solution:

(1) Let's first assume that $f = 0$ outside $[-1,1]$. Applying a result form Problem set there exists a sequence of step functions g_n such that for any R, $g_n \to g$ *uniformly* on $[0,1)$. By passing to a subsequence we can arrange that $\sup_{[-1,1]} |g_n(x) - g_{n-1}(x)| < 2^{-n}$. If f_n is an absolutly summable series of step functions converging a.e. to f we can replace it by $f_n \chi([-1,1])$ as discussed above, and still have the same conclusion. Thus, from the uniform convergence of g_n,

$$g_n(x) \sum_{k=1}^{n} f_k(x) \to g(x) f(x) \text{ a.e. on } \mathbb{R}. \tag{8.53}$$

So define $h_1 = g_1 f_1$, $h_n = g_n(x) \sum\limits_{k=1}^{n} f_k(x) - g_{n-1}(x) \sum\limits_{k=1}^{n-1} f_k(x)$. This series of step functions converges to $gf(x)$ almost everywhere and since

(8.54)
$$|h_n| \le A|f_n(x)| + 2^{-n} \sum_{k<n} |f_k(x)|, \ \sum_n \int |h_n| \le A \sum_n \int |f_n| + 2 \sum_n \int |f_n| < \infty$$

it is absolutely summable. Here A is a bound for $|g_n|$ independent of n. Thus $gf \in \mathcal{L}^1(\mathbb{R})$ under the assumption that $f = 0$ outside $[0,1)$ and

(8.55)
$$\int |gf| \le \sup |g| \int |f|$$

follows from the limiting argument. Now we can apply this argument to f_p which is the restriction of p to the interval $[p, p+1)$, for each $p \in \mathbb{Z}$. Then we get gf as the limit a.e. of the absolutely summable series gf_p where (8.55) provides the absolute summablitly since

(8.56)
$$\sum_p \int |gf_p| \le \sup |g| \sum_p \int_{[p,p+1)} |f| < \infty.$$

Thus, $gf \in \mathcal{L}^1(\mathbb{R})$ by a theorem in class and

(8.57)
$$\int |gf| \le \sup |g| \int |f|.$$

(2) If $f \in L^1[(0,1])$ has a representative f' then $G(x,\cdot)f'(\cdot) \in \mathcal{L}^1([0,1))$ so

(8.58)
$$F(x) = \int_{[0,1]} G(x,\cdot)f(\cdot) \in \mathbb{C}$$

is well-defined, since it is indpendent of the choice of f', changing by a null function if f' is changed by a null function.

(3) Now by the uniform continuity of continuous functions on a compact metric space such as $S = [0,1] \times [0,1]$ given $\delta > 0$ there exist $\epsilon > 0$ such that

(8.59)
$$\sup_{y\in[0,1]} |G(x,y) - G(x',y)| < \delta \text{ if } |x - x'| < \epsilon.$$

Then if $|x - x'| < \epsilon$,

(8.60)
$$|F(x) - F(x')| = |\int_{[0,1]} (G(x,\cdot) - G(x',\cdot))f(\cdot)| \le \delta \int |f|.$$

Thus $F \in \mathcal{C}([0,1])$ is a continuous function on $[0,1]$. Moreover the map $f \longmapsto F$ is linear and

(8.61)
$$\sup_{[0,1]} |F| \le \sup_S |G| \int_{[0,1]} ||f|$$

which is the desired boundedness, or continuity, of the map

(8.62)
$$I : L^1([0,1]) \longrightarrow \mathcal{C}([0,1]),\ F(f)(x) = \int G(x,\cdot)f(\cdot),$$
$$\|I(f)\|_{\sup} \le \sup |G| \|f\|_{L^1}.$$