18.112 Functions of a Complex Variable
Fall 2008

Lecture 8: Line Integrals

(Text 101-108)

Remarks on Lecture 8

The following rule (integration by substitution) is often useful.

Theorem 1 *Let $w = \varphi(z)$ be a holomorphic function on a region Ω. Let γ be a curve in Ω, then*

$$\int_{\varphi(\gamma)} f(w)\, dw = \int_\gamma f(\varphi(z))\varphi'(z)\, dz.$$

Proof: Let γ be given by

$$\gamma \ : \ z(t), \ \alpha \le t \le \beta.$$

Then $\varphi(\gamma)$ is given by

$$\varphi \ : \ w(t) = \varphi(z(t)), \ \alpha \le t \le \beta.$$

Then LHS equals

$$\int_\alpha^\beta f(w(t))w'(t)\, dt,$$

and RHS equals

$$\int_\alpha^\beta f(\varphi(z(t)))\varphi'(z(t))z'(t)\, dt = \int_\alpha^\beta f(w(t))w'(t)\, dt.$$

Q.E.D.

Exercise 3 on page 108.

We have

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right),$$

so the problem is reduced to computing

$$\int_\gamma \frac{1}{z-1}\,dz \qquad \text{and} \qquad \int_\gamma \frac{1}{z+1}\,dz,$$

where $\gamma \;:\; |z| = 2$ is the circle.

Since $\log(z+1)$ is holomorphic in the region $\mathbb{C} \setminus I_\epsilon$, where I_ϵ is the shown wedge with vertex -1 and opening of angle ϵ. In this region

$$\frac{d(\log(z+1))}{dz} = \frac{1}{z+1}.$$

Letting $\epsilon \to 0$ we deduce from Theorem p.107 middle

$$\int_{\gamma+\Gamma} \frac{1}{z+1}\,dz = 0.$$

Hence

$$\int_\gamma \frac{1}{z+1}\,dz = 2\pi i.$$

Similarly,

$$\int_\gamma \frac{dz}{z-1} = 2\pi i$$

using the circle $|z-1| = 1$. Consequently,

$$\int_\gamma \frac{dz}{z^2-1} = 0.$$

The result is obvious from our substitution theorem because if

$$\varphi(z) = -z,$$

then

$$\varphi(\gamma) = \gamma \qquad \text{(including orientation)}.$$

So

$$\int_\gamma \frac{dz}{z^2 - 1} = - \int_\gamma \frac{dz}{z^2 - 1}.$$

More generally we have

Theorem 2 *Let R be a rational function on $\mathbb{C}$. Then*

$$\int_\gamma R(z^2)\ dz = 0$$

for every circle γ around the origin provided $R(z^2) \neq 0$ on γ.