Chapter 6

Geometric Invariant Theory

Lecture 31

Lie Groups

Goof references for this material: Abraham-Marsden, Foundations of Mechanics (2nd edition) and Ana Canas p. 128

Let G be a lie group. Denote by $\mathfrak{g}$ the Lie algebra of G which is T_eG, with the lie bracket operation.

Definition. The exponential is a map $\exp : \mathfrak{g} \to G$ with the following properties

(a) $\mathbb{R} \to G$, $t \mapsto \exp tv$ is a lie group homomorphism.

(b)
$$\frac{d}{dt} \exp tv \Big|_{t=0} = v \in T_eG = \mathfrak{g}$$

Example. $G = GL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det(A) \neq 0\}$. Then $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = M_{n\times n}(\mathbb{R})$ and $[A, B] = AB - BA$ and
$$\exp A = \sum \frac{A^i}{i!}$$

Example. G a compact connected abelian Lie group. Then the lie algebra is $\mathfrak{g}$ with $[,] \equiv 0$. $\mathfrak{g}$ is a vector space, i.e. an abelian lie group in its own right. Then the exponential map $\exp : \mathfrak{g} \to G$ is a surjective lie group homomorphism.

Let $\mathbb{Z}_G = \ker \exp$ be called the Group lattice of G, then $G = \mathfrak{g}/\mathbb{Z}_G$, by the first isomorphism theorem.

For instance, take $G = (S^1)^n = T^n$, then $\mathfrak{g} = \mathbb{R}^n$, $\exp : \mathbb{R}^n \to T^n$ is given by $(t_1, \ldots, t_n) \mapsto (e^{it_1}, \ldots, e^{it_n})$. Then $\mathbb{Z}_G = 2\pi\mathbb{Z}^n$ and $G \cong \mathbb{R}^n/2\pi\mathbb{Z}^n$.

Group actions

Let M be a manifold.

Definition. An action of G on M is a group homomorphism
$$\tau : G \to Diff(M)$$
where τ is smooth if $ev : G \times M \to M$, $(g, m) \to \tau_g(m)$ is smooth.

Definition. Then infinitesimal action of $\mathfrak{g}$ on M
$$d\tau : \mathfrak{g} \to Vect(M) \qquad v \in \mathfrak{g} \mapsto v_M$$
is given by
$$\tau(\exp tv) = exp(-tv_M)$$

Theorem. *$d\tau$ is a morphism of lie algebras.*

Given $p \in M$ denote
$$G_p = \{g \in G, \tau_g(p) = p\}$$
This is the **isotropy group of** p of the **stabilizer of** p. Then
$$\operatorname{Lie} G_p = \{v \in \mathfrak{g} \mid v_m(p) = 0\}$$

Definition. The orbit of G through p is
$$G \circ p = \{\tau_g(p) \mid g \in G\}$$

This is an immersed submanifold of M, and its tangent space is given by $T_p(G \circ p) = \mathfrak{g}/\mathfrak{g}_p$.

The orbit space of τ is M/G = the set of all orbits, or equivalently $M/\sim$ where $p, q \in M$ and $p \sim q$ iff $p = \tau_g(q)$ for some $g \in G$.

We can topologize this space, by the projection
$$\pi : M \to M/G \qquad p \mapsto G \circ p$$
and define the topology of M/G by $U \subset M/G$ is open if and only if $\pi^{-1}(U)$ is open (i.e. assign M/G the weakest topology that makes π continuous). This, however, can be a nasty topological space.

Example. $M = \mathbb{R}$, $G = (\mathbb{R}^+, \times)$. And τ maps t to multiplication by t. Then M/G is composed of 3 points, $\pi(0), \pi(1)$ and $\pi(-1)$, but the set $\{\pi(1), \pi(-1)\}$ is not closed.

Definition. The action τ is **free** if $G_p = \{e\}$ for all p (e the identity).

Definition. The action τ is **locally free** if $\mathfrak{g} = \{0\}$ for all p (this happens if and only if G_p is discrete).

Definition. τ is a proper action if the map $G \times M \to M \times M$ given by $(g, m) \mapsto (m, \tau_g(m))$ is a proper map.

Theorem. *If τ is free and proper then $M/G$is a differentiable manifold and $\pi : M \to M/G$ is a smooth fibration.*

Proof. (Sketch) S a slice of a G-orbit through pi.e, S is a submanifold of M of codim $= \dim G$, with $S \cap G \circ p = \{p\}$, $T_pS \oplus T_pG \circ p = T_pM$. Its not hard to construct such slices.

Then look at the map $G \times S \to M$, $(g, s) \to \tau_g(s)$. This is locally a diffeomorphism at (e, p) and group invariance implies that it is locally a diffeomorphism on $G \times \{p\}$. So it maps a neighborhood W of $G \times \{p\}$ diffeomorphically onto an open set $O \subseteq M$.

Properness insures that $W = G \times U_0$ where $(U_0, x_1, \dots, x_n)$ is a coordinate patch on S centered at p.

Let $U = O/G \cong U_0$ and $(U, x_1, \dots, x_n)$ is a coordinate patch on M/G.

We claim that any two such coordinate patches are compatible (Maybe add a figure here?) □

Definition. G is a **complex Lie group** if G is a complex manifold and the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are holomorphic.

Example. (a) $G = GL(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det A \neq 0\}$. And the lie algebra is $M_n(\mathbb{C}) = \mathfrak{gl}(n, \mathbb{C})$.

(b) $\mathbb{C}^* = \mathbb{C} - \{0\}$.

(c) Complex Tori. For instance $T^n_{\mathbb{C}} = (\mathbb{C}^*)^n$.

Definition. An action τ of G on M is **holomorphic** if
$$ev : G \times M \to M$$
is holomorphic.

In particular for $g \in G$, $\tau_g : M \to M$ is a biholomorphism and the G-orbits
$$G \circ p$$
are complex submanifolds of G.

Theorem. *If τ is free and proper the orbit space M/G is a complex manifold and the fibration $\pi : M \to M/G$ is a holomorphic fiber mapping.*

Proof. Imitate the proof above with S being a holomorphic slice of $G \circ p$ at p. □

Symplectic Manifolds and Hamiltonian G-actions

Let G be a connected Lie group and M, ω a symplectic manifold. An action, τ of G on M is symplectic if $\tau_g^* \omega = \omega$ for all g,i.e. the τ_g are symplectomorphisms.

Thus if $v \in \mathfrak{g}$

$$\tau(\exp tv)^* \omega = \omega = \exp(-tv_M)^* \omega$$

Then

$$\frac{d}{dt} \exp(-tv_M)^* \omega \Big|_{t=0} = L_{v_M} \omega = 0.$$

This implies that

$$\iota(v_M) d\omega + d\iota(v_M)\omega = d\iota(v_M)\omega = 0$$

so $\iota_{v_M} \omega$ is closed.

Definition. τ is a **Hamiltonian action** if for all $v \in \mathfrak{g}$, $\iota(v_M)\omega$ is exact.

The Moment Map

Choose a basis $v^1, \ldots, v^n$ of $\mathfrak{g}$ and let $v_1^*, \ldots, v_n^*$ be a dual basis of $\mathfrak{g}^*$.

If τ is hamiltonian then $\iota(v_M^i)\omega = d\phi^i$, where $\phi^i \in C^\infty(M)$.

Definition. The map $\Phi : M \to \mathfrak{g}^*$ defined by

$$\Phi = \sum \phi^i v_i^*$$

is called the **moment map**

Remarks

(a) Note that for every $v \in \mathfrak{g}$,

$$\iota(v_M)\omega = d\phi^v \qquad \text{where } \phi^v = \langle \Phi, v \rangle$$

(b) Φ is only well defined up to an additive constant $c \in \mathfrak{g}^*$.

(c) If M is compact one can normalize this constant by requiring that

$$\int_M \phi^i \frac{\omega^n}{n!} = 0$$

(d) Another normalization: If $p \in M^G$, i.e. if $G_p = G$, then one can require that $phi^i(p) = 0$ for $i = 1, \ldots, n$, then $\Phi(p) = 0$.