Lecture 35

Let G be a compact connected Lie group and $n = \dim G$, with Lie algebra $\mathfrak{g}$. We have a group lattice $\mathbb{Z}_G \subset \mathfrak{g}$, and the dual $\mathbb{Z}_G^* \subset \mathfrak{g}^*$ the weight lattice. Then $G = \mathfrak{g}/\mathbb{Z}_G$. We can define $\exp : \mathfrak{g} \to \mathfrak{g}/\mathbb{Z}_G$.

Take elements $\alpha_i \in \mathbb{Z}_G^*$, $i = 1, \dots, d$ then we get a representation $\tau : G \to GL(d, \mathbb{C})$ given by

$$\tau(\exp v)z = (e^{\sqrt{-1}\alpha_1(v)}z_1, \dots, e^{\sqrt{-1}\alpha_d}z_d).$$

We can think of τ as an action. As such it preserves the Kaehler form

$$\omega = \sqrt{-1}\sum dz_i \wedge d\bar{z}_i$$

In fact, τ is Hamiltonian with momen t map

$$\Phi : \mathbb{C}^d \to \mathfrak{g}^*, \qquad \Phi(z) = \sum |z_i|^2 \alpha_i$$

Note that $\alpha_1, \dots, \alpha_d$ are polarized if and only if there exists a $v \in \mathfrak{g}$ such that $\alpha_i(v) > 0$ for all i.

Theorem. *α_is are polarized if and only if Φ, the moment map, is proper.*

What are the regular values of Φ?
Let

$$\mathbb{R}_+^d = \{(t_1, \dots, t_d) \in \mathbb{R}^d, t_i \geq 0\}$$

and take $I \subseteq \{1, \dots, d\}$.

Notation. $\mathbb{R}_I^d = \{t \in \mathbb{R}^d, t_i \neq 0 \Leftrightarrow i \in I\}$

Consider the following maps: $L : \mathbb{R}^d \to \mathfrak{g}^*$ given by

$$t \mapsto \sum t_i \alpha_i$$

and $\gamma : \mathbb{C}^d \to \mathbb{R}_+^d$ given by

$$z \mapsto (|z_1|^2, \dots, |z_n|^2).$$

Then for any $a \in \mathfrak{g}^*$, let $\Delta_a = L^{-1}(a) \cap \mathbb{R}_+^d$. Then $\Phi = L \circ \gamma$, so $z \in \Phi^{-1}(a)$ if and only if $\gamma(z) \in \Delta_a$.

Suppose that the α_is are polarized. Then Δ_a is a compact convex set, and in fact it is a **convex polytope**

Definition. The **index set of a polytope** is defined to be

$$\mathcal{I}_{\Delta_a} = \{I \mid \mathbb{R}_I^d \cap \Delta_a \neq 0\}$$

The **faces of the polytope** Δ_a are the sets

$$\Delta_I = \Delta_a \cap \mathbb{R}_I^d, \qquad I \in \mathcal{I}_\Delta$$

Theorem (1). *Let $a \in \mathfrak{g}^*$. Then*

(a) a is a regular value of Φ if and only if for every $I \in \mathcal{I}_{\Delta_a}$

$$span_{\mathbb{R}}\{a_i, i \in I\} = \mathfrak{g}^*$$

(b) G acts freely on $\Phi^{-1}(a)$ if and only if for all $I \in \mathcal{I}_{\Delta_a}$

$$span_{\mathbb{Z}}\{a_i, i \in I\} = \mathbb{Z}_G^*$$

$\mathcal{I}_\Delta$ is partially order by inclusion, i.e. $I_1 < I_2$ if $I_1 \subseteq I_2$. $I \in \mathcal{I}_\Delta$ is minimal iff the corresponding face Δ_I is a vertex of Δ_a, i.e. $\Delta_I = \{v_I\}$ where v_I is a vertex of Δ_a.

Theorem (2). *(a) a is a regular value of the moment map Φ if and only if for every vertex v_I of Δ_a, $\alpha_i, i \in I$ are a basis of $\mathfrak{g}^*$.*

(b) G acts freely on $\Phi^{-1}(a)$ if andonly if for every vertex v_I of Δ_a, $\alpha_i, i \in I$ are a lattice basis for $\mathbb{Z}_G^*$.

Proof. In **Theorem 1** it suffices to check a) and b) for the minimal elements I of $\mathcal{I}_\Delta$.

Check that a) of Thm. 1 implies b) of Thm 2. So we just have to check a) of Thm. 2.

Let $\Delta_I = \{v_I\}$, where I is a minimal element of $\mathcal{I}_\Delta$. By Thm 1., $\text{span}\{\alpha_i, i \in I\} = \mathfrak{g}^*$. Suppose α_is are not a basis, then there exist c_i so that

$$\sum_{i \in I} c_i \alpha_i = 0$$

Now, $v_I = (t_1, \dots, t_d)$, $t_i > 0$ for $i \in I$ and $t_i = 0$ for $i \notin I$. Define $(s_1, \dots, s_d) \in \Delta_a$ by

$$s_i = \begin{cases} t_i + \epsilon c_i & i \in I \\ 0 & i \notin I \end{cases}$$

□

Then $L(s) = a$, $s \in \Delta_I$, so this contradicts that Δ_I is a singular point.

Notation. $\Delta \in \mathbb{R}^d$ a convex polytope, $v, v' \in Vert(\Delta)$. Then v and v' are **adjacent** if they lie on a common edge of Δ.

Definition. An m-dimensional polytope Δ is **simple** if for every vertex v there are exactly m vertices adjacent to it.

[Next time we'll show that a is a regular value of Φ iff Δ_a is simple]

Example. A tetrahedron or a cube in $\mathbb{R}^3$. A pyramid is not simple.

$\Phi : \mathbb{C}^d \to \mathfrak{g}^*$, and a a regular value. G acts freely on $Z_a = \Phi^{-1}(a)$. Then we can form the symplectic quotient $M_a = \Phi^{-1}(a)/G$, which is a compact Kaehler manifold. We want to compute the de Rham and Dolbeault cohomology groups, $H^*_{DR}(M_a), H^*_{Do}(M_a)$. To compute the de Rham cohomology we're going to use Morse Theory.

A Digression on Morse Theory

Let M^m be a compact C^∞ manifold and let $f : M \to \mathbb{R}$ be a smooth function.

$p \in \textbf{Crit}(f)$ if and only if $df_p = 0$ (by definition). For any $p \in \text{Crit}(f)$ we have the Hessian $d^2 f_p$ a quadratic form on T_p. Let $(U, x_1, \dots, x_n)$ be a coordinate patch centered at p. Then

$$f(x) = c + \sum a_{ij} x_i x_j + O(x^3) = d^2 f_p + O(x^3)$$

and p is called non-degenerate if $d^2 f_p$ is non-degenerate. If p is a non-degenerate critical point, then p is isolated.

Definition. f is Morse if all $p \in \text{Crit}(f)$ are non-degenerate, which implies that

$$\#\text{Crit}(f) < \infty$$

Definition. $p \in \text{Crit}(f)$ then $\text{ind} p = \text{ind}\, d^2 f_p$, i.e. if

$$d^2 f_p = -(x_1^2 + \cdots + x_k^2) + x_{k+1}^2 + \cdots + x_m^2$$

then $\text{ind}\, d^2 f_p = k$.

Theorem. *Let $f : M \to \mathbb{R}$ be a Morse function with the property tat $\text{ind}\, p$ is even for all $p \in Crit(f)$. Then*

$$H^{2k+1}(M) = 0 \qquad H^{2k}(M) = \{p \in Crit(f), \text{ind}\, p = 2k\}$$

Back to Symplectic Reduction

Again, we're talking about the moment map $\Phi : \mathbb{C}^d \to \mathfrak{g}^*$, with a a regular value of Φ. G acts freely on Z_a and let $M_a = Z_a/G$. Then we have the following diagram:

$$\begin{tikzcd} Z_a \arrow[r, "i"] \arrow[d, "\pi"'] & \mathbb{C}^n \\ M_a & \end{tikzcd}$$

and the mapping $\gamma : \mathbb{C}^d \to \mathbb{R}^d_+$, $z \mapsto (|z_1|^2, \ldots, |z_d|^2)$. γ is G-invariant.

This implies that there exists $\psi : M_a \to \mathbb{R}^d_+$ with the property that $\psi \circ \pi = \gamma \circ i$. Moreover $\gamma : Z_a \to \Delta_a$. So $\psi : M_a \to \Delta_a$, Δ_a is called the moment polytope.

Now take $\xi \in \mathbb{R}^d$ and let $f : M_a \to \mathbb{R}$ be $f(p) = \langle \psi(p), \xi \rangle$, i.e. $\pi^* f = i^* f_0$ where

$$f_0(z) = \sum \xi_i |z_i|^2$$

Theorem (Main Theorem). *Assume for $v, v' \in Vert(\Delta_a)$, v, v' adjacent that*

$$\langle v - v', \xi \rangle \neq 0$$

then

(a) $f : M_a \to \mathbb{R}$ is Morse

(b) $\psi : M_a \to \Delta_a$ maps Crit(f) bijectively onto $Vert(\Delta_a)$.

(c) For $p \in$ Crit(f) and v the corresponding vertex let $v_1, \ldots, v_m$ be the vertices adjacent to v. Then

$$\frac{ind_p}{2} = \#\{v_i \mid \langle v_i - v, \xi \rangle < 0\} := ind_v \xi$$

Corollary. *$H^{2k+1}(M_a) = 0$ then*

$$b_k = H^{2k}(M_a) = \#\{v \in Vert(\Delta_a), ind_v \xi = k\}$$

that is, b_k is independent of ξ.