MINIMAL GRAPHS

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For Lipschitz $u : B := B_1^n(0) \to \mathbf{R}$ and $\Omega \overset{open}{\subset} B$, define

$$\mathcal{A}(u;\Omega) = \text{Area}\,(\text{graph}\, u|_\Omega) = \int_\Omega \sqrt{1+|Du|^2},$$

where, by 'area', we mean n-dimensional Hausdorff measure. The notation $\mathcal{A}(u)$ simply means Area $(\text{graph}\, u)$. It can be established (firstly for C^1 functions using integration by parts and duality and then by approximating a Lipschitz function in the L^1 norm by a sequence of C^1 functions) that for any Lipschitz $u : B \to \mathbf{R}$, we have

$$\mathcal{A}(u) = \sup_{\substack{g \in C_c^1(B,\mathbf{R}^{n+1}):\\ \|g\|_\infty \le 1}} \int_B g_{n+1} + u \operatorname{div}(g_1, ..., g_n). \tag{1}$$

From here one can easily deduce that $\mathcal{A}$ is *strictly* convex on $\mathcal{L}$ and lower semicontinuous with respect to weak L^1 convergence.

Given $f \in C^2(\partial B)$, if $u \in C^2(\overline{B})$ is such that $u|_{\partial B} = f$ and $\mathcal{A}(u) \le \mathcal{A}(\tilde{u})$ for any $\tilde{u} \in C^2(\overline{B})$ with $\tilde{u}|_{\partial B} = f$, then u solves the Dirichlet problem for the Minimal Surface Equation, *i.e.*

$$\mathcal{M}u := \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \ \text{ in } B$$
$$u = f \ \text{ on } \partial B$$

Write $\mathcal{L}_L$ for the set of all Lipschitz functions $u : B \to \mathbf{R}$ with $u|_{\partial B} = f$ and $\operatorname{Lip} u \le L$ and write $a_L := \inf_{\tilde{u} \in \mathcal{L}_L} \mathcal{A}(\tilde{u})$. The symbols $\mathcal{L}$ and a are defined similarly, but without the condition that $\operatorname{Lip} u \le L$.

Lemma 1. *If $\mathcal{L}_L \neq \emptyset$, then exists $u_L \in \mathcal{L}_L$ such that $\mathcal{A}(u_L) = a_L$ and such that if $\operatorname{Lip} u_L < L$, then $\mathcal{A}(u_L) = a$*

Proof. Since $\mathcal{L}_L \neq \emptyset$, we can take a sequence $\{u^j\}_{j=1}^\infty \in \mathcal{L}_L$ with $\mathcal{A}(u^j) \to a_L$ as $j \to \infty$. The fact that $\operatorname{Lip} u^j \le L$ for every j means that $\{u^j\}_{j=1}^\infty$ is equicontinuous and for sufficiently large j, this sequence must also be uniformly bounded (why?). Therefore the Arzela – Ascoli theorem implies that there exists $v \in \mathcal{L}_L$ and a subsequence $\{j'\}$ of $\{j\}$ such that $u^{j'} \to v$ uniformly as $j \to \infty$. Setting $u_L := v$, the first claim follows from the lower semicontinuity of $\mathcal{A}$ with respect to weak L^1 convergence.

Now given $\tilde{u} \in \mathcal{L}$ observe that $(tu_L + (1-t)\tilde{u})|_{\partial B} = f$ for all $t \in (0,1)$ and that $\operatorname{Lip}(tu_L + (1-t)\tilde{u}) \le k$ for sufficiently small $t > 0$. Thus for sufficiently small $t > 0$ we have $(tu_L + (1-t)\tilde{u}) \in \mathcal{L}_L$ whence $\mathcal{A}(u_L) \le \mathcal{A}(tu_L + (1-t)\tilde{u})$. The convexity of $\mathcal{A}$ then implies that $\mathcal{A}(u_L) \le t\mathcal{A}(u_L) + (1-t)\mathcal{A}(\tilde{u})$, from which the second claim follows. $\square$

For $\Omega \overset{open}{\subset} B$, write $\mathcal{L}_L(\Omega)$ for the set of all Lipschitz functions $u : \Omega \to \mathbf{R}$ with $\operatorname{Lip} u \leq L$.

We will need more terminology for our next result: A function $u \in \mathcal{L}_L(\Omega)$ is called a *supersolution* (resp. *subsolution*) in $\mathcal{L}_L(\Omega)$ if for every $\tilde{u} \in \mathcal{L}_L(\Omega)$ with $\tilde{u}|_{\partial\Omega} = u|_{\partial\Omega}$ and $\tilde{u} \geq u$ ($\tilde{u} \leq u$) we have that $\mathcal{A}(\tilde{u};\Omega) \geq \mathcal{A}(u;\Omega)$. In particular, a minimizer in $\mathcal{L}_L(\Omega)$ (*i.e.* a function the area of the graph of which is smallest among competitors with the same boundary values) is both a super- and a subsolution.

Lemma 2. *For $\Omega \subset B$, let v and u be super- and subsolutions in $\mathcal{L}_L(\Omega)$. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.*

Proof. Suppose for the sake of contradiction, that $\mathcal{S} = \{x \in \Omega : v(x) < u(x)\} \neq \emptyset$ and write $m = \min\{u, v\}$. Since $m \in \mathcal{L}_L(\Omega)$, $m|_{\partial\Omega} = u|_{\partial\Omega}$ and $m \geq u$ in Ω, the fact that u is a subsolution in $\mathcal{L}_L(\Omega)$ means that $\mathcal{A}(u;\Omega) \leq \mathcal{A}(m;\Omega)$, which implies that $\mathcal{A}(u;\mathcal{S}) \leq \mathcal{A}(v;\mathcal{S})$. Now, the strict convexity of the area functional tells us that

$$\mathcal{A}(\tfrac{1}{2}u + \tfrac{1}{2}v;\mathcal{S}) < \tfrac{1}{2}\mathcal{A}(u;\mathcal{S}) + \tfrac{1}{2}\mathcal{A}(v;\mathcal{S}) \leq \mathcal{A}(v;\mathcal{S}). \tag{2}$$

But, since $w = \max\{\frac{1}{2}u + \frac{1}{2}v, v\}$ satisfies $w \in \mathcal{L}_L(\Omega)$, $w|_{\partial\Omega} = v|_{\partial\Omega}$ and $w \geq v$ in Ω, the fact that v is a supersolution means that $\mathcal{A}(v;\Omega) \leq \mathcal{A}(w;\Omega)$, which implies that $\mathcal{A}(v;\mathcal{S}) \leq \mathcal{A}(\frac{1}{2}u + \frac{1}{2}v;\mathcal{S})$, contradicting the strict inequality in (2). □

Corollary 3. *Let u and v be respectively a subsolution and supersolution in $\mathcal{L}_L(\Omega)$. Then*

$$\sup_{x\in\Omega}[u(x) - v(x)] = \sup_{x\in\partial\Omega}[u(x) - v(x)].$$

Proof. For every $\alpha \in \mathbf{R}$, the function $v(x) + \alpha$ is also a supersolution and for any $x \in \partial\Omega$, we clearly have $u(x) \leq v(x) + \sup_{y\in\partial\Omega}[u(y) - v(y)]$. The result now follows from the previous lemma. □

We can now reduce our goal of bounding the Lipschitz constant to a boundary estimate:

Lemma 4. *With u_L as in Lemma 1 we have*

$$\operatorname{Lip} u_L = \sup_{x\in B,\ y\in\partial B} \frac{|u_L(x) - u_L(y)|}{|x - y|} \tag{3}$$

Proof. Let x_1, x_2 be distinct points in $B = B_1(0)$. Both u and the function $x \mapsto u(x + x_2 - x_1)$ minimize area in $\mathcal{L}_L(B_1(0) \cap B_1(x_1 - x_2))$ and so both functions are supersolutions and subsolutions in $\mathcal{L}_L(B_1(0) \cap B_1(x_1 - x_2))$. Applying corollary 3 in the domain $B_1(0) \cap B_1(x_1 - x_2)$ and with $u(x + x_2 - x_1)$ in place of $v(x)$ implies that there exists $z \in \partial(B_1(0) \cap B_1(x_1 - x_2))$ for which

$$|u(x_1) - u(x_2)| \leq |u(z) - u(z + x_2 - x_1)|.$$

At least one of z and $z + x_2 - x_1$ belongs to ∂B, so on dividing by $|x_2 - x_1|$, we get the result. □

Proposition 5 (Existence of Barriers)**.** *Given $f \in C^2(\overline{B})$, there exist constants c_1, c_2 and $r > 0$ (depending only on f and n) such that $v : B_1(0) \setminus B_{1-r}(0) \to \mathbf{R}$ given by $v(x) = f(x) + c_1 \log(1 + c_2 \operatorname{dist}(x, \partial B))$ has the following properties.*

1. *$v^+|_{\partial B} = f|_{\partial B}$.*
2. *$v^+|_{\partial B_{1-r}(0)} \geq \sup_{\partial B} f$.*

(3) v^+ is a supersolution in $\mathcal{L}(B_1(0) \setminus B_{1-r}(0))$.

Sketch of Proof. Try $v^+(x) = f(x) + \psi(d(x))$, for some $\psi \in C^2([0, 1-r])$ satisfying $\psi(0) = 0$, $\psi' \geq 1$, $\psi'' < 0$ and $\psi(1-r) \geq 2\sup_\Omega |f|$. Such a v^+ will automatically satisfy the first two conditions above. Then one computes $(1+|Dv|^2)^{3/2}\mathcal{M}v^+$ and using the fact that $\Delta \mathrm{dist}(x, \partial B) < 0$ near ∂B deduces that $(1+|Dv|^2)^{3/2}\mathcal{M}v^+ \leq \psi'' + C\psi'^2$. Choosing ψ as above suffices to ensure that this last expression is less than zero, which implies that v^+ is a supersolution.

The function v here is called an 'upper barrier'. If one take the negative of the function obtained by applying this Proposition with $-f$ in place of f, one gets a function v^- satisfying

(1) $v^-|_{\partial B} = f|_{\partial B}$.
(2) $v^-|_{\partial B_{1-r}(0)} \leq \inf_{\partial B} f$.
(3) v^- is a subsolution in $\mathcal{L}(B_1(0) \setminus B_{1-r}(0))$.

Such a function is called a 'lower barrier'.

In more general domains, the geometry of the boundary of the domain plays an important role: For a C^2 domain Ω, we would first need to show that $d \in C^2(\{x : d(x) < r\} \cap \Omega)$ for sufficiently small $r > 0$ and then we would need to assume that $\partial\Omega$ were mean convex because this is what implies in general that $\Delta d \leq 0$ in a neighbourhood of the boundary of Ω such as $\{x : d(x) < r\} \cap \Omega$. See [GT01, §14.6]. Armed with these facts, one can similarly prove the existence of functions $v^\pm$ that satisfy analogous conclusions in the domain $\{x : d(x) < r\} \cap \Omega$. If the boundary of Ω is not mean convex, then there is smooth boundary data for which the Dirichlet problem cannot be solved.

Theorem 6. *Given $f \in C^2(\partial B)$, there exists $u \in \mathcal{L}$ with $\mathcal{A}(u) \leq \mathcal{A}(\tilde{u})$ for all $\tilde{u} \in \mathcal{L}$*

Proof. By solving the Dirichlet problem for the Laplacian with boundary values f, we can ensure that $\mathcal{L}_L$ is non-empty for some sufficiently large L and assume that $f \in C^2(\overline{B})$. Let us also pick $L > \mathrm{Lip}\, v^+$.

By applying proposition 5 with $-f$ in place of f, we get another function $w : B_1(0) \setminus B_{1-r}(0) \to \mathbf{R}$ (a 'lower barrier') for which

$$v^-(x) \leq \inf_{\partial B} f \leq u(x) \leq \sup_{\partial B} f \leq v^+. \tag{4}$$

on $\partial B_{1-r}(0)$. Since u_L is minimizing in $\mathcal{L}_L(B_1(0) \setminus B_{1-r}(0)$ and since $-w$ is a subsolution in $B_1(0) \setminus B_{1-r}(0)$, Lemma 2 implies that $v^- \leq u \leq v^+$ on $B_1(0) \setminus B_{1-r}(0)$. Then, using the fact that $u = -w = v$ on $\partial B_1(0)$, we have

$$|u(x) - u(y)| \leq (\mathrm{Lip}\, v)|x - y| \tag{5}$$

for every $x \in B_1(0) \setminus B_{1-r}(0)$, $y \in \partial B$. On the other hand, if $x \in B_{1-r}(0)$, we have that

$$|u(x) - u(y)| \leq \max\left\{\sup_{\partial B} f - u(y), u(y) - \inf_{\partial B} f\right\} \leq \mathrm{Lip}\, v,$$

(using (2) in the defining properties of $v^\pm$) which shows that (5) holds for all $x \in B$. Then, by lemma 4 we have $\mathrm{Lip}\, u \leq \mathrm{Lip}\, v < L$ and so the conclusion follows from Lemma 1. ∎

Regularity of Lipschitz Weak Solutions

In this section we will use notions of weak derivatives and Sobolev spaces to discuss the regularity properties of the Lipschitz function the existence of which is asserted by Theorem 6. We will focus on showing that $u \in C^{1,\alpha}_{loc}(B)$. The reader is directed to [Eva09, Chapter 5] or [GT01, Chapter 7] for an introduction to weak derivatives and Sobolev spaces.

The function u the existence of which is asserted by Theorem 6 satisfies

$$0 = \frac{d}{dt}\Big|_{t=0} \mathcal{A}(u + t\eta), \tag{6}$$

from which a short calculation gives that

$$0 = \int_B \frac{Du \cdot D\eta}{\sqrt{1+|Du|^2}}, \tag{7}$$

for every $\varphi \in C^\infty_c(\Omega)$. Any function $u \in W^{1,2}(B)$ that satisfies (7) is called a weak solution of the minimal surface equation.

Differentiating the Equation. Suppose that we were able to show that $u \in W^{2,2}_{loc}(B)$. Now if we set $w = D_k u$ for some k, by assumption $w \in W^{1,2}(B)$. Then, replacing η by $D_k\eta$ as a test function in (7), we have

$$\int_B \frac{D_i u}{\sqrt{1+|Du|^2}} D_i(D_k\eta) = 0.$$

Integrating by parts, we obtain

$$\int_B \tilde{a}_{ij}(x) D_j w D_i \eta = 0,$$

where

$$\tilde{a}_{ij}(x) = \frac{\delta_{ij}}{\sqrt{1+|Du|^2}} - \frac{D_i u D_j u}{(1+|Du|^2)^{3/2}}.$$

Using the fact that u is Lipschitz, it is easy to check that these coefficients are bounded and uniformly elliptic (*i.e.* there exist $c, C > 0$ such that $c|\xi|^2 \leq \tilde{a}_{ij}(x)\xi_i\xi_j \leq C|\xi|^2 \;\forall\; x \in B_1$ and $\xi \in \mathbf{R}^n$). Thus on any smaller ball $B' \subset B$, we have that w is indeed a $W^{1,2}$ weak solution to a uniformly elliptic 2nd order PDE in divergence form with bounded, measurable coefficients. From here, De Giorgi-Nash-Moser Theory tells us that w is locally Hölder continuous. So, provided that $u \in W^{2,2}_{loc}(B)$, we have that $u \in C^{1,\alpha}_{loc}(B)$. Let us establish the former.

Lemma 7. *Let u be a Lipschitz weak solution to the minimal surface equation in the domain Ω. Then $u \in W^{2,2}_{loc}(\Omega)$.*

Proof. To highlight the salient points of the proof define $F : \mathbf{R}^n \to \mathbf{R}^n$ by $F(p) = p(1+|p|^2)^{-1/2}$ and observe that

(1) $\frac{\partial F^i}{\partial p_j} = \frac{\partial F^j}{\partial p_i}$ for all $1 \leq i, j \leq n$.
(2) $c|\xi|^2 \leq \frac{\partial F^i}{\partial p_j}(Du(x))\xi_i\xi_j \leq C|\xi|^2 \;\forall\; x \in B_1$ and $\xi \in \mathbf{R}^n$.
(3) $0 = \int_\Omega F_i(Du) D_i \eta$ for all $\eta \in W^{1,2}_0(\Omega)$.

Fix a direction e_k in the standard orthonormal basis. For $\varphi \in C^\infty_c(\Omega)$ and $h > 0$ such that $|h| < \frac{1}{2}\mathrm{dist}(\partial\Omega, \operatorname{supp}\varphi)$, set $\eta = \Delta^{-h}(\varphi^2 \Delta^h u)$ where

$$(\Delta^h f)(x) := (\Delta^h_k f)(x) := \frac{f(x+he_k) - f(x)}{h}. \tag{8}$$

Note that the following 'integration by parts' formula for these difference quotients holds: $\int f\Delta^{-h}g = -\int \Delta^h fg$, so on plugging η in as a test function we get that

$$\int_\Omega \Delta^h F_i(Du)D_i(\varphi^2\Delta^h u) = 0. \tag{9}$$

Using the fundamental theorem of calculus followed by the chain rule

$$\Delta^h F_i(Du) = h^{-1}\int_0^1 \frac{d}{dt}F_i(Du + th\Delta^h Du)dt \tag{10}$$

$$= \int_0^1 \frac{\partial F_i}{\partial p_j}(Du + th\Delta_h Du)dt\, D_j\Delta^h u \tag{11}$$

and therefore on writing $\theta_{ij} := \int_0^1 \frac{\partial F_i}{\partial p_j}(Du + th\Delta^h Du)dt$ and $v := \Delta^h u$ we have that (9) reads:

$$\int_\Omega \theta_{ij}D_jvD_i(\varphi^2 v) = 0. \tag{12}$$

Now fix $\Omega' \Subset \Omega$ and insist that $\varphi \equiv 1$ on Ω' and $0 \le \varphi \le 1$ on Ω. Consider $\Omega'' \Subset \Omega'$ and pick h such that $|h| < \operatorname{dist}(\Omega'', \partial\Omega')$. One can check (using (2)) that there exists constants c, C such that

$$c|\xi|^2 \le \theta_{ij}(x)\xi_i\xi_j \le C|\xi|^2.$$

for all $x \in \Omega'$. We now compute:

$$\begin{aligned}
c\int_{\Omega''}|Dv|^2 &\le c\int_{\Omega'}|D(\varphi v)|^2 \text{ because } \varphi \equiv 1 \text{ on } \Omega'' \\
&\le \int_{\Omega'}\theta_{ij}D_j(v\varphi)D_i(v\varphi) \\
&= \int_{\Omega'}\theta_{ij}\left[D_j\varphi D_i\varphi v^2 + 2\varphi vD_jvD_i\varphi + \varphi^2 D_jvD_iv\right] \text{ using } \theta_{ij} = \theta_{ji}, \\
&= \int_{\Omega'}\theta_{ij}D_j\varphi D_i\varphi v^2 + \int_\Omega \theta^{ij}D_jvD_i(\varphi^2 v) \text{ using the product rule,} \\
&= \int_{\Omega'}\theta_{ij}D_j\varphi D_i\varphi v^2 \text{ from (12)} \\
&\le C\int_{\Omega'}v^2|D\varphi|^2 \\
&\le K,
\end{aligned}$$

where the final bound is achieved using the fact that u is Lipschitz and the constant K depends on $\operatorname{dist}(\Omega', \partial\Omega)$. From here we can apply the standard results about difference quotients, *e.g.* [Eva09, Theorem 3, §5.8.2] or [GT01, Lemma 7.24] to deduce that $u \in W^{2,2}_{loc}(\Omega')$, hence $u \in W^{2,2}_{loc}(\Omega)$. $\square$

References

[Eva09] L.C. Evans, *Partial differential equations (graduate studies in mathematics, vol. 19)*, Instructor (2009).

[GT01] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer Verlag, 2001.

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