18.156 Differential Analysis II
Lecture 21

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Lecture 21: 1 April 2015

21.1 Finishing the Sierpinski Theorem

Recall from our wishful thinking that we wished in part to bound $|\widehat{Pf}(n)|$ for $n \in \mathbb{Z}^2$. We also had $\widehat{Pf}(n) = \widehat{f}(n)$, so the following estimate is useful.

Proposition 1. *If $f = \chi_{B(R)}$ in $\mathbb{R}^2$, then*

$$|\widehat{f}(w)| \lesssim CR^{1/2}|w|^{-3/2}.$$

Proof Sketch. We have that by rotational invariance,

$$\begin{aligned} |\widehat{f}(w)| &= \left| \int_{B(R)} e^{iw\cdot x} dx \right| \\ &= \left| \int_{B(R)} e^{i|w|x_1} dx_1 dx_2 \right| \\ &= 2\left| \int_{-R}^{R} (R^2 - x^2)^{1/2} e^{i|w|x} dx \right| \end{aligned}$$

We first integrate by parts with $u = (R^2 - x^2)^{1/2}$ and $dv = e^{iwx}dx$ to obtain

$$|\widehat{f}(w)| = \frac{1}{|w|} \left| \int_{-R}^{R} (R^2 - x^2)^{1/2} x e^{iwx} \, dx \right|.$$

If we apply the triangle inequality immediately, we only obtain a bound of about Rw^{-1}. Instead, we would like something better. The idea is to break up the domain $[-R, R]$ into two regions, one an interval of the form $[-S, S]$ and the other the set with $S < |x| < R$. On the former region, we can integrate by parts again, and on the outer region we can use the triangle inequality, and then optimize the result by changing S. It turns out that taking $S = 1/|w|$ is the right choice, but either way, we get the estimate in the proposition. ☐

Remark 2. Also, of course, $|\widehat{f}(w)| \leq \pi R^2$ by the triangle inequality directly, and this is a better estimate for small R.

Now, we have $N(R) = Pf(0)$, and our wishful thinking would have us hope that this is $\pi R^2 + \sum_{n\in\mathbb{Z}^2\setminus\{0\}} \widehat{f}(n)$. But this sum isn't even absolutely summable given our estimate $|\widehat{f}(n)| \lesssim R^{1/2}|n|^{-3/2}$. Instead, we need to do something slightly different. Our trick is to approximate f by smooth functions so that we get better estimates and convergent sums. In order to do this, consider η a smooth bump function with support in $B(1)$ and $\int \eta = 1$. Set $f_{R,\epsilon} := \chi_{B(R)} * \eta_\epsilon$, where $\eta_\epsilon(x) = \epsilon^{-2}\eta(x/\epsilon)$ still has integral 1, but the support becomes more and more localized to 0 as $\epsilon \to 0$. For now, we will fix R and simply write $f_\epsilon := f_{R,\epsilon}$, and we will go back after the next proposition. The function f_ϵ is now smooth and compactly supported, and so we get the convergence properties needed in the sums we are looking at, and in particular, we can prove the following:

Proposition 3. $|Pf_\epsilon(0) - \pi R^2| \leq CR^{1/2}w^{-1/2}$.

Proof. Because f_ϵ is smooth, any sum we write in this proof will converge, as the reader can verify. Recall that the Fourier transform commutes with convolution, so we have

$$Pf_\epsilon(0) = \sum_{n\in\mathbb{Z}^2} \widehat{Pf_\epsilon}(n) = \sum_{n\in\mathbb{Z}^2} \widehat{f}_\epsilon(n) = \sum_{n\in\mathbb{Z}^2} \widehat{f}(n)\widehat{\eta}_\epsilon(n).$$

The term with $n = 0$ is just πR^2, and so the left hand side of the inequality of our proposition is just

$$\left|\sum_{n\in\mathbb{Z}^2-\{0\}} \widehat{f}(n)\widehat{\eta}_\epsilon(n)\right|.$$

Now, since η is compactly supported, we get $|\widehat{\eta}_\epsilon(n)| \lesssim (1+|n|\epsilon)^{-10000}$, and so applying the triangle inequality to the sum at hand and using our estimates for $|\widehat{f}(n)|$ and $|\widehat{\eta}_\epsilon(n)|$, we find that

$$|Pf_\epsilon(0) - \pi R^2| \lesssim R^{1/2} \sum_{n\neq 0} |n|^{-3/2}(1+|n|\epsilon)^{-10000} \sim R^{1/2} \sum_{0<|n|<\epsilon^{-1}} |n|^{-3/2} \sim R^{1/2}w^{-1/2}.$$

□

Proof of Sierpinski Theorem. Note that $f_\epsilon \geq 1$ on $B(R-\epsilon)$, and so we have $N(R) \leq Pf_{R+\epsilon,\epsilon}(0)$, and so we find

$$\begin{aligned} N(R) &\leq \pi(R+\epsilon)^2 + CR^{1/2}\epsilon^{-1/2} \\ &= \pi R^2 + CR\epsilon + CR^{1/2}\epsilon^{-1/2} \\ &\leq \pi R^2 + CR^{2/3} \end{aligned}$$

where the last inequality comes from choosing $\epsilon = R^{-1/3}$ (which is the best possible choice by AM-GM). There is a similar lower bound, and so we find $|E(R)| \lesssim R^{2/3}$ as desired. □