Lecture 23

May 11^{th}, 2004

$\mathbf{L^p}$ Theory

Take f any measurable function on a domain $\Omega \subseteq \mathbb{R}^n$ and define the *distribution function of f* $\mu_f(t) := |\{x \in \Omega : |f(x)| > t\}|$. We use alternatively $|\cdot|$ and $\lambda(\cdot)$ to denote the *Lebesgue measure*.

Proposition. *Assume $f \in L^p(\Omega)$ for some $p > 0$.*

$$I) \quad \mu_f(t) \leq t^{-p} \int_\Omega |f|^p d\mathbf{x}.$$

$$II) \quad \int_\Omega |f|^p d\mathbf{x} = p \int_0^\infty t^{p-1} \mu_f(t) dt.$$

In order for the second equation to make sense we need the distribution function to be measurable and indeed it is as f itself is.

Proof. First

$$\int_\Omega |f|^p d\mathbf{x} \geq \int_{\{f>t\}} |f|^p d\mathbf{x} \geq t^p \lambda(\{x : \; f(x) > t\}) = t^p \mu_f(t).$$

Second, assume first $p = 1$. By Fubini's Theorem one can interchange order of integration in

$$\int_\Omega |f| = \int_\Omega \int_0^{|f(x)|} dt d\mathbf{x} = \int_0^\infty \int_\Omega \mathbb{I}_{\{x \in \Omega: \; f(x) > t\}} d\mathbf{x} dt = \int_0^\infty \mu_f(t) dt.$$

For general p

$$\mu_{f^p}(t) = |\{x : \; f^p(x) > t\}| = |\{x : \; f(x) > \sqrt[p]{t}\}| = \mu_f(\sqrt[p]{t}) =$$

and so

$$p\int_0^\infty t^{p-1}\mu_f(t)dt = \int_0^\infty \mu_{f^p}(t^p)d(t^p) = \int_\Omega |f|^p d\mathbf{x}. \qquad \blacksquare$$

$\mathcal{M}$arcinkiewicz Interpolation Theorem. *Let $1 \le q < r < \infty$ and let $T: L^q(\Omega) \cap L^r(\Omega) \longrightarrow L^q(\Omega) \cap L^r(\Omega)$ be a linear map. Suppose there exist constants T_1, T_2 such that*

$$\forall\, f \in L^q(\Omega) \cap L^r(\Omega) \qquad \mu_{Tf}(t) \le \Big(\frac{T_1||f||_{L^q(\Omega)}}{t}\Big)^q, \quad \mu_{Tf}(t) \le \Big(\frac{T_2||f||_{L^r(\Omega)}}{t}\Big)^r, \quad \forall\, t > 0.$$

Then for any exponent in between $q < p < r$, T can be extended to a map $L^p(\Omega) \longrightarrow L^p(\Omega)$ for all $f \in L^q(\Omega) \cap L^p(\Omega)$. And moreover,

$$||Tf||_{L^p(\Omega)} \le \Big[\frac{p}{q-p}(2T_1)^q + \frac{p}{r-p}(2T_2)^r\Big]^{\frac{1}{p}} ||f||_{L^p(\Omega)}.$$

Otherwise stated: weak (q,q) & weak (r,r) $\implies$ strong (p,p) $p \in (q,r)$, though not for the endpoints, the constants blow-up there (we say an operator is *strong* (p_1,p_2) if it maps functions in L^{p_1} to functions in L^{p_2}. We say it is *weak* (p_1,p_2) if its domain is in L^{p_1} and its distribution function satisfies the first inequality in the assumptions above with q replaced by p_2).

Proof. Take $f \in L^q(\Omega) \cap L^r(\Omega)$, and let $s > 0$. Let

$$f_1 := \begin{cases} f(x) & |f(x)| > s \\ 0 & |f(x)| \le s \end{cases}$$

$$f_2 := \begin{cases} 0 & |f(x)| > s \\ f(x) & |f(x)| \le s \end{cases}$$

indeed one notices that $f = f_1 + f_2$. The trick will be to let this splitting of f vary by letting s itself vary. So $|Tf| \le |Tf_1| + |Tf_2|$. If $Tf(x) > t$ at some point $x \in \Omega$ then either $Tf_1 > t/2$ or $Tf_2 > t/2$. This translates into

$$\mu_{Tf}(t) \le \mu_{Tf_1}(t/2) + \mu_{Tf_2}(t/2)$$
$$\le \Big(\frac{T_1}{t/2}\Big)^q \int_\Omega |f_1|^q + \Big(\frac{T_2}{t/2}\Big)^r \int_\Omega |f_2|^r.$$

We choose the smaller exponent q for the terms where f is large (f_1) and larger one r for where f is small (f_2), intuitively. This will make sense in a moment when it will be clear how this guarantees that our two integrals — with different integration domains — are finite. By the Proposition we have

$$\int_\Omega |Tf|^p d\mathbf{x} = p\int_0^\infty t^{p-1}\mu_{Tf}(t)dt.$$

and once we substitute in the above inequality we get

$$\int_\Omega |Tf|^p d\mathbf{x} \le p\int_0^\infty t^{p-1}\Big[\Big(\frac{T_1}{t/2}\Big)^q \int_\Omega |f_1|^q + \Big(\frac{T_2}{t/2}\Big)^r \int_\Omega |f_2|^r\Big]dt$$
$$= p(2T_1)^q \int_0^\infty \Big(\int_{\{|f|>s\}} |f_1|^q\Big) t^{p-1-q}dt + p(2T_2)^r \int_0^\infty \Big(\int_{\{|f|\le s\}} |f_2|^r\Big) t^{p-1-r}dt.$$

We chose $s > 0$ arbitrary in the above construction of f_i. In particular we may let it vary. This is a neat trick. We set $s = t$ to get

$$p(2T_1)^q \int_0^\infty \Big(\int_{\{|f|>s\}} |f|^q\Big) s^{p-1-q}ds + p(2T_2)^r \int_0^\infty \Big(\int_{\{|f|\le s\}} |f|^r\Big) s^{p-1-r}ds$$
$$= p(2T_1)^q \int_\Omega |f|^q d\mathbf{x} \int_0^{|f|} s^{p-1-q}ds + p(2T_2)^r \int_\Omega |f|^r d\mathbf{x} \int_{|f|}^\infty s^{p-1-r}ds$$
$$= (2T_1)^q \frac{p}{q-p}\int_\Omega |f|^p + (2T_2)^r \frac{p}{r-p}\int_\Omega |f|^p.$$

Altogether

$$\int_\Omega |Tf|^p d\mathbf{x} \le \Big[\frac{p}{q-p}(2T_1)^q + \frac{p}{r-p}(2T_2)^r\Big] \cdot \int_\Omega |f|^p.$$

■

Remark. In Gilbarg-Trudinger, p.229, a different constant is achieved which is slightly stronger than ours (as can be seen using the AM-GM Inequality). This is done by introducing an additional constant A, letting $t = As$ and later choosing A appropriately.

Back to the Newtonian Potential

We defined the *Newtonian Potential* of f

$$\omega \equiv Nf := \int_\Omega \Gamma(x-y)f(y)d\mathbf{y} = \frac{1}{n(2-n)\omega_n}\int_\Omega \frac{1}{|x-y|^{n-2}}d\mathbf{y}.$$

Claim. **(Young's Inequality)** $N : L^p(\Omega) \longrightarrow L^p(\Omega)$. *Moreover continuously so–* $\exists\ C$ *such that* $||Nf||_{L^p(\Omega)} \leq C||f||_{L^p(\Omega)}$.

Remark. For $p = 2$ we proved in the past much more: $\Delta(Nf) = f \in L^2(\Omega) \;\Rightarrow\; Nf \in W^{2,2}(\Omega)$. Also our previous estimates on the Newtonian Potential can actually be made to extend our Claim to $W^{1,p}(\Omega)$ regularity. These estimates can not give though $W^{2,p}(\Omega)$ estimates (see the beginning of the next Lecture).

Proof.

$$\begin{aligned}
\omega := \Gamma \star f &= \int_\Omega \Gamma(x-y)f(y)d\mathbf{y} \\
&= \int_\Omega f(y)\Gamma(x-y)^{\frac{1}{p}}\Gamma(x-y)^{1-\frac{1}{p}}d\mathbf{y} \\
&\leq \Big\{\int_\Omega |f(y)^p\Gamma(x-y)|d\mathbf{y}\Big\}^{\frac{1}{p}}\Big\{\int_\Omega |\Gamma(x-y)|d\mathbf{y}\Big\}^{1-\frac{1}{p}} \\
&\leq C\cdot\Big\{\int_\Omega |f(y)^p\Gamma(x-y)|d\mathbf{y}\Big\}^{\frac{1}{p}}.
\end{aligned}$$

since $\Gamma(x-y) \sim \dfrac{1}{|x-y|^{n-1}}$ and therefore is integrable over $\mathbb{R}^n$. Therefore we have an upper bound on ω^p which we can integrate

$$\int_\Omega \omega^p d\mathbf{x} \leq \int_\Omega C^p \{ \int_\Omega |f(y)^p \Gamma(x-y)| d\mathbf{y} \} d\mathbf{x}$$

$$= C^p \int_\Omega \int_\Omega |f(y)|^p |\Gamma(x-y)| d\mathbf{x} d\mathbf{y}$$

$$= C^p \int_\Omega |f(y)|^p \big(\int_\Omega |\Gamma(x-y)| d\mathbf{x} \big) d\mathbf{y}$$

$$\leq \tilde{C} \int_\Omega |f(y)|^p d\mathbf{y}.$$

where we applied Fubini's Theorem. ■

Theorem. *Let $f \in L^p(\Omega)$ for some $1 < p < \infty$ and let $\omega = Nf$ be the Newtonian Potential of f. Then $\omega \in W^{2,p}(\Omega)$ and $\Delta w = f$ a.e. and*

$$||D^2 w||_{L^p(\Omega)} \leq c(n, p, \Omega) \cdot ||f||_{L^p(\Omega)}.$$

For $p = 2$ we have even

$$\int_{\mathbb{R}^n} |D^2 \omega|^2 = \int_\Omega f^2.$$

Proof. We prove just for $p = 2$, leaving the hard work for the next and last lecture. First we assume $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. From long time ago: $f \in \mathcal{C}_0^\infty(\mathbb{R}^n) \;\Rightarrow\; \omega \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\Delta\omega = f$ (Hölder Theory for the Newtonian Potential).

Let $B := B_R$ a ball containing supp$f$ $\quad\Rightarrow$

$$\int_{B_R} (\mathrm{D}\omega)^2 = \int_{B_R} f^2 = \int_\Omega f^2 \tag{1}$$

We embark now on our main computation

$$\int_{B_R} |\mathrm{D}^2\omega|^2 = \int_{B_R} \mathrm{D}_{ij}\omega\mathrm{D}_{ij}\omega \text{ (summation)} \quad = -\int_{B_R} \mathrm{D}_j(\mathrm{D}_{ij}\omega)\mathrm{D}_i\omega + \int_{\partial B_R} \mathrm{D}_{ij}\omega\mathrm{D}_i\omega\nu_j d\theta$$

$$= -\int_{B_R} \mathrm{D}_i(\mathrm{D}_{jj}\omega)\mathrm{D}_i\omega + \int_{\partial B_R} \mathrm{D}_{ij}\omega\mathrm{D}_i\omega\nu_j d\theta$$

$$= -\int_{B_R} \mathrm{D}_i(\Delta\omega)\mathrm{D}_i\omega + \int_{\partial B_R} \frac{\partial}{\partial\nu}\mathrm{D}\omega\mathrm{D}\omega d\theta$$

$$= \int_{B_R} (\Delta\omega)^2 - \int_{\partial B_R} \Delta\omega \cdot \frac{\partial}{\partial\nu}\omega d\theta + \int_{\partial B_R} \frac{\partial}{\partial\nu}\mathrm{D}\omega \cdot \mathrm{D}\omega d\theta$$

$$= \int_{B_R} (\Delta\omega)^2 + \int_{\partial B_R} \frac{\partial}{\partial\nu}\mathrm{D}\omega \cdot \mathrm{D}\omega d\theta.$$

The last equality results from our assumption that f vanishes on ∂B, i.e has compact support inside Ω. Now since f is smooth

$$\mathrm{D}_i\omega(x) = \int_\Omega \mathrm{D}_i\Gamma(x-y)f(y)d\mathbf{y} \leq \frac{C}{R^{n-1}},$$

$$\mathrm{D}_{ij}\omega(x) = \int_\Omega \mathrm{D}_{ij}\Gamma(x-y)f(y)d\mathbf{y} \leq \frac{C}{R^{n}}.$$

Therefore as we let $R \longrightarrow \infty$, the second term - which is integrated only over the sphere of radius R in $\mathbb{R}^n$ – tends to 0. Then we have in the limit the desired result (after substituting (1) for the RHS).

Now if $f \in L^2(\Omega)$, approximate it by functions $f_m \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (possible by the density argument used in the past: $\overline{\mathcal{C}_0^\infty(\Omega)} = L^2(\Omega)$) such that $f_m \xrightarrow{L^2(\Omega)} f$. From the Claim above $||Nf||_{L^p(\Omega)} \leq C||f||_{L^p(\Omega)}$, hence $||N(f_i - f_j)||_{L^p(\Omega)} \leq C||f_i - f_j||_{L^p(\Omega)}$, from which $\omega_m \equiv Nf_m \xrightarrow{L^2(\Omega)} Nf \equiv \omega$. Now $\Delta\omega_j = f_j$ and by the $\mathcal{C}_0^\infty(\Omega)$ case applied to the Dirichlet Problem $\Delta(\omega_i - \omega_j) = f_i - f_j$

$$\int_{\mathbb{R}^n} |\mathrm{D}^2(\omega_i - \omega_j)|^2 = \int_\Omega |f_i - f_j|^2.$$

As the RHS tends to 0 for i, j large we have that $\{D^2\omega_m\}$ converges in $L^2(\Omega)$, i.e $\{\omega_m\}$ converges in $W^{2,2}(\Omega)$. Since we already know its limit is $\omega \in L^2(\Omega)$ we conclude that in fact $\omega \in W^{2,2}(\Omega)$! ∎