Lecture 1

Mean Value Theorem

Theorem 1 *Suppose* $\Omega \subset \mathbb{R}^n$, $u \in C^2(\Omega)$, $\Delta u = 0$ *in* Ω, *and* $B = B(y, R) \subset\subset \Omega$, *then*

$$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} uds = \frac{1}{\omega_n R^n} \int_B udx$$

Proof: By Green's formula, for $r \in (0, R)$, $\int_{\partial B_r} \frac{\partial u}{\partial \nu} ds = \int_{B_r} \Delta u dx = 0$. Thus

$$\begin{aligned} 0 = \int_{\partial B_r} \frac{\partial u}{\partial \nu} ds &= \int_{\partial B_r} \frac{\partial u}{\partial r}(y + r\omega) ds \\ &= r^{n-1} \int_{S^{n-1}} \frac{\partial u}{\partial r}(y + r\omega) d\omega \\ &= r^{n-1} \frac{\partial}{\partial r} \int_{S^{n-1}} u(y + r\omega) d\omega \\ &= r^{n-1} \frac{\partial}{\partial r}(r^{1-n} \int_{\partial B_r} uds) \end{aligned}$$

$\Longrightarrow \frac{1}{r^{n-1}} \int_{\partial B_r} uds = const$ for any r.

But we also have

$$n\omega_n u_{min}(B_r) \leq \frac{1}{r^{n-1}} \int_{\partial B_r} uds \leq n\omega_n u_{max}(B_r),$$

taking limit as $r \to \infty$, we get for any r

$$u(y) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r} uds.$$

Integral it, we get the solid mean value thm. ■

Remark 1 *We have* $\triangle u \geq 0 \Longrightarrow u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} uds$, *and we call such u sub-harmonic, i.e. u lies below hamonic function sharing the same boundary values.*

Also we have $\triangle u \leq 0 \Longrightarrow u(y) \geq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} uds$ *and we call u super-harmonic.*

Application: Maximum principle and uniqueness.

Theorem 2 $\Omega \subset \mathbb{R}^n, u \in C^2(\Omega), \Delta u \geq 0$, *If* $\exists p \in \Omega$ *s.t.*

$$u(p) = \max_{\Omega} u,$$

then u *is constant.*

Proof: Let

$$M = \sup_{\Omega} u, \qquad \Omega_M = \{x \in \Omega | u(x) = M\}.$$

Ω_M is not empty because $p \in M$, Ω_M is closed by continuity, Ω_M is open by mean value inequality. Thus $\Omega_M = M$, i.e. u is constant function. ■

Corollary 1 $u \in C^2(\Omega) \bigcap C^0(\overline{\Omega})$, $\Delta u = 0$, *then if* Ω *bounded, we have*

$$\inf_{\partial\Omega} u \leq \sup_{\partial\Omega}, \qquad x \in \Omega.$$

Corollary 2 $u, v \in C^2(\Omega) \bigcap C^0(\Omega), \Delta u = \Delta v$ *in* Ω, $u = v$ *on* $\partial\Omega \Longrightarrow u \equiv v$ *on* $\partial\Omega$.

Corollary 3 $\Delta u \geq 0, \Delta v = 0, u \equiv v$ *on* $\partial\Omega \Longrightarrow u \leq v$ *in* Ω. *(Hence "subharmonic")*

Application: Harnack Inequality.

Theorem 3 *Suppose* Ω *domain,* $u \in C^2(\Omega), \Delta u = 0, \Omega' \subset\subset \Omega, u \geq 0$ *in* Ω, *then* $\exists$ *constant* $C = C(n, \Omega, \Omega')$ *s.t.*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u.$$

Proof: Let $y \in \Omega', B(y, 4R) \subset \Omega$. Take $x_1, x_2 \in B(y, R)$, we have

$$u(x_1) = \frac{1}{\omega_n R^n} \int_{B(x_1,R)} u dx \leq \frac{1}{\omega_n R^n} \int_{B(y,2R)} u dx,$$

$$u(x_2) = \frac{1}{\omega_n (3R)^n} \int_{B(x_2,3R)} u dx \geq \frac{1}{\omega_n (3R)^n} \int_{B(y,2R)} u dx,$$

$$\Longrightarrow u(x_1) \leq 3^n u(x_2),$$

$$\Longrightarrow \sup_{B(y,R)} \leq 3^n \inf_{B(y,R)}.$$

Choose R little enough s.t. $B(y, 4R) \subset \Omega$ for $\forall y \in \Omega'$. Let $x_1, x_2 \in \overline{\Omega'}$ s.t. to be maximal and minimal point of u in $\overline{\Omega'}$ respectively. We can cover Ω' by N balls of radius R since Ω' is compact, so we have

$$\sup_{\Omega'} u \leq u(x_1) \leq 3^n u(x_1') \leq \cdots \leq 3^{nN} \inf_{\Omega'} u.$$

This completes our proof. ■

Remark 2 *1. A Harnack inequality implies C^α regularity for $0 < \alpha < 1$.*
2. A positive (or more generally bounded above or below) harmonic function on $\mathbb{R}^n$ is constant.

A Priori Estimate for harmonic function.

Theorem 4 *$u \in C^\infty, \Delta u = 0, \Omega' \subset \Omega$. Then for multi-index α, there exists constant $C = C(n, \alpha, \Omega, \Omega')$ s.t.*

$$\sup_{\Omega'} |D^\alpha u| \leq C \sup_{\Omega} |u|.$$

Proof: Since $\frac{\partial}{\partial x_i}\Delta = \Delta \frac{\partial}{\partial x_i}$, Du is also harmonic. So by mean value theorem and divergence theorems, we have for $B(y, R) \subset \Omega$,

$$Du(y) = \frac{1}{\omega_n R^n} \int_{B(y,R)} Du dx = \frac{1}{\omega_n R^n} \int_{\partial B} u \overrightarrow{\nu} ds$$

$$\Longrightarrow |Du(y)| \leq \frac{n}{R} \sup_{\partial B} |u|$$

$$\Longrightarrow |Du(y)| \leq \frac{n}{d(y, \partial\Omega)} \sup_{\Omega} |u|.$$

By induction, we get the stated estimate for higher order derivatives. ■

Remark 3 *We can weaken the assumptions to $u \in C^2(\Omega)$: $u \in C^2(\Omega)$ and $\Delta u = 0 \Longrightarrow u$ analytic. We will do this next time.*

Green's Representation Formula.

Suppose Ω is C^1 domain, $u, v \in C^2(\overline{\Omega})$.
Green's 1^{st} identity:

$$\int_\Omega v \Delta u dx + \int_\Omega Du \cdot Dv dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} ds.$$

Green's 2^{nd} identity:

$$\int_\Omega (v \Delta u - u \Delta v) dx = \int_{\partial\Omega} (v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}) ds.$$

Find solution for Laplacian:

$$\Gamma(x) = \begin{cases} \frac{1}{n(2-n)\omega_n} |x|^{2-n} & , \quad n > 2, \\ \frac{1}{2\pi} \log |x| & , \quad n = 2. \end{cases}$$

Note that away from origin, $\Delta\Gamma(x) = 0$.

Theorem 5 *Suppose $u \in C^2(\overline{\Omega})$, then for $y \in \Omega$, we have*

$$u(y) = \int_{\partial\Omega} (u\frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y)\frac{\partial u}{\partial \nu})d\sigma + \int_{\Omega} \Gamma(x-y)\Delta u dx.$$

Proof: Take ρ small enough s.t. $B_\rho = B_\rho(y) \subset \Omega$. Apply Green's 2^{nd} formula to u and $v(x) = \Gamma(x-y)$, which is harmonic in $\Omega \setminus \{y\}$, on the domain $\Omega \setminus B_\rho$, we get

$$\int_{\Omega\setminus B_\rho} \Gamma(x-y)\Delta u dx = \int_{\partial\Omega} (\Gamma(x-y)\frac{\partial u}{\partial \nu} - u\frac{\partial \Gamma}{\partial \nu}(x-y))d\sigma + \int_{\partial B_rho} (\Gamma(x-y\frac{\partial u}{\partial \nu} - u\frac{\partial \Gamma}{\partial \nu}(x-y)))d\sigma.$$

Let $\rho \to 0$, notice that as $\rho \to 0$

$$|\int_{\partial B_\rho} \Gamma(x-y)\frac{\partial u}{\partial \nu}d\sigma| \le \Gamma(\rho)\sup_{B_\rho}|Du| n\omega_n\rho^{n-1} \to 0,$$

$$\int_{\partial B_rho} u\frac{\partial \Gamma}{\partial \nu}(x-y)d\sigma = -\Gamma'(\rho)\int_{\partial B_\rho} u d\sigma = \frac{-1}{n\omega_n\rho^{n-1}}\int_{\partial B_\rho} u d\sigma \to -u(y),$$

thus we get the Green's Representation Formula. ■

Application of Green's Formula:

Theorem 6 *Let $B = B_R(0)$ and φ is continuous function on ∂B. Then*

$$u(x) = \begin{cases} \frac{R^2 - |x^2|}{n\omega_n R}\int_{\partial B}\frac{\varphi(y)}{|x-y|^n}ds & , \quad x \in B, \\ \varphi(x) & , \quad x \in \partial B. \end{cases}$$

belongs to $C^2(B) \cap C^0(\overline{B})$ and satisfies $\Delta u = 0$ in B.