18.303 Midterm, Fall 2014

Each problem has **equal weight**. You have 55 minutes.

Problem 1: Adjoints (25 points)

Consider the operator $\hat{A}u = \frac{d^2}{dx^2}(cu)$ on the domain $\Omega = [0, L]$ with Dirichlet boundaries $u|_{\partial\Omega} = 0$, where $c(x) > 0$. **Show** that $\hat{A} = \hat{A}^*$ for an appropriate choice of inner product $\langle u, v \rangle$.

Problem 2: Green (25 points)

Consider the operator $\hat{A} = -\nabla^2$ in 2d, where the domain is the entire x, y plane. In cylindrical (r, ϕ) coordinates, we can write

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}.$$

We want to solve for the Green's function $G(\mathbf{x}, \mathbf{x}') = g(|\mathbf{x} - \mathbf{x}'|)$ of $\hat{A}$, reduced to a function $g(r)$ by the symmetry of the problem. By looking at $r > 0$, one can quickly show that $g(r) = c \ln r$ for some unknown constant c. This g defines a regular distribution, if we just set $g(0) = 0$, despite the fact that ln blows up [it is an "integrable" singularity in 2d: the integral exists for any test function $\psi(x, y)$]. **Find** c.

Problem 3: Waves (25 points)

In class, we re-wrote the wave equation $u'' = \ddot{u}$ in the form $\hat{D}\mathbf{w} = \partial\mathbf{w}/\partial t$, where $\mathbf{w} = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\hat{D} = \begin{pmatrix} & \partial/\partial x \\ \partial/\partial x & \end{pmatrix}$. We showed that $\hat{D}^* = -\hat{D}$ (anti-Hermitian) for appropriate boundary conditions (let's say $u = 0$ on the boundaries) for the obvious inner product $\langle \mathbf{w}, \mathbf{w}' \rangle = \int_\Omega \mathbf{w}^*\mathbf{w}'$, and hence $\hat{D}$ has purely imaginary eigenvalues $\lambda = -i\omega$ (oscillating solutions), and $\|\mathbf{w}\|^2 = \langle \mathbf{w}, \mathbf{w} \rangle$ was a conserved "energy."

Consider a new operator:

$$\hat{D}_\sigma = \begin{pmatrix} -\sigma & \partial/\partial x \\ \partial/\partial x & -\sigma \end{pmatrix}$$

where $\sigma(x) > 0$.

(a) **Show** that $\|\mathbf{w}\|^2$ is *not* conserved; it is _______ in time. Hint: consider $\partial\langle \mathbf{w}, \mathbf{w} \rangle/\partial t$ as in class.

(b) **Show** that if $\mathbf{w}_n$ is an eigenfunction, i.e. $\hat{D}_\sigma \mathbf{w}_n = \lambda_n \mathbf{w}_n$, then the *real* part of λ is negative (hence the eigensolutions are ________ in time). Hint: consider $\langle \mathbf{w}_n, (\hat{D}_\sigma + \hat{D}_\sigma^*)\mathbf{w}_n \rangle$.

Problem 4: Discrete Waves (25 points)

Consider the operator

$$\hat{D}_\sigma = \begin{pmatrix} -\sigma & \partial/\partial x \\ \partial/\partial x & -\sigma \end{pmatrix}$$

from the previous problem, acting on $\mathbf{w} = \begin{pmatrix} u \\ v \end{pmatrix}$. For $\sigma = 0$, we discretized this in class via $u_m^n \approx u(m\Delta x, n\Delta t)$ and $v_{m+0.5}^{n+0.5}$ (a staggered grid in space and time):

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \frac{v_{m+0.5}^{n+0.5} - v_{m-0.5}^{n+0.5}}{\Delta x},$$

$$\frac{v_{m+0.5}^{n+0.5} - v_{m+0.5}^{n-0.5}}{\Delta t} = \frac{u_{m+1}^n - u_m^n}{\Delta x}.$$

Write down a center-difference (second-order accurate) scheme for $\partial\mathbf{w}/\partial t = \hat{D}_\sigma\mathbf{w}$. For simplicity, take σ to be a constant (independent of x or t), and solve for:

$$u_m^{n+1} = ?$$
$$v_{m+0.5}^{n+0.5} = ?$$