Test 1

18.303 Linear Partial Differential Equations

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Oct. 12, 2006

1 Rules

You may only use pencils, pens, erasers, and straight edges. No calculators, notes, books or other aides are permitted. Scrap paper will be provided.

Be sure to show a few key intermediate steps when deriving results - answers only will not get full marks.

2 Given

You may assume the eigenvalues of the Sturm-Liouville problem

$$\begin{aligned} X'' + \lambda X &= 0, \qquad 0 < x < 1 \\ X'(0) &= 0 \qquad X(1) = 0 \end{aligned}$$

are $\lambda_n = (2n-1)^2 \frac{\pi^2}{4}$ and $X_n(x) = \cos\left(\frac{2n-1}{2}\pi x\right)$, for $n = 1, 2, ...$, without derivation.

You may assume the following orthogonality conditions for m, n positive integers:

$$\int_0^1 \sin\left(\frac{2m-1}{2}\pi x\right) \sin\left(\frac{2n-1}{2}\pi x\right) dx = \begin{cases} 1/2, & m = n, \\ 0, & m \neq n. \end{cases}$$

$$\int_0^1 \cos\left(\frac{2m-1}{2}\pi x\right) \cos\left(\frac{2n-1}{2}\pi x\right) dx = \begin{cases} 1/2, & m = n, \\ 0, & m \neq n. \end{cases}$$

You may use the following inequality

$$|\cos 3\alpha| \leq 3 |\cos \alpha|, \qquad \text{all reals } \alpha.$$

3 Questions

Total points: 30

Consider the following heat problem in dimensionless variables

$$\begin{aligned} u_t &= u_{xx} + bx^2, \qquad 0 < x < 1, \qquad t > 0 \\ \frac{\partial u}{\partial x}(0,t) &= 0, \qquad u(1,t) = 1, \qquad t > 0 \\ u(x,0) &= u_0 \qquad 0 < x < 1, \end{aligned}$$

where $b > 0$ and $u_0 > 0$ are constants. This is the heat equation with a source, where the rod is insulated at $x = 0$ and kept at 1 degree at $x = 1$.

(a) [3 points] Derive the steady-state (equilibrium) solution

$$u_E(x) = \frac{b}{12}\left(1 - x^4\right) + 1$$

It is NOT sufficient to simply verify that the solution works.

(b) [3 points] Using $u_E(x)$, transform the given heat problem for $u(x,t)$ into the following problem for a function $v(x,t)$:

$$\begin{aligned} v_t &= v_{xx}, \qquad 0 < x < 1, \qquad t > 0 \\ \frac{\partial v}{\partial x}(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0 \\ v(x,0) &= f(x) \qquad 0 < x < 1. \end{aligned}$$

where $f(x)$ will be determined by the transformation. Show your work, which involves writing $v = u - u_E$ and using the information from u and u_E to derive the problem for v. State $f(x)$ in terms of u_0, b and x.

(c) [11 points] Derive the solution

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-(2n-1)^2\pi^2 t/4} \cos\left(\frac{2n-1}{2}\pi x\right)$$

and derive equations for A_n in terms of $f(x)$. Be sure to give the intermediate steps: separate variables, write down problems and solve for $X(x)$ (using information from the Given section), solve for $T_n(t)$, put things together, impose the IC. Use orthogonality of $\cos\left((2n-1)\frac{\pi}{2}x\right)$ (see Given section) to find A_n in terms of $f(x)$. Substitute for $f(x)$ from part (b). You may use (without proof) the following integrals, for any integer n,

$$\int_0^1 \cos\left(\frac{2n-1}{2}\pi x\right) dx = \frac{2(-1)^{n+1}}{(2n-1)\pi}$$

$$\int \left(1-x^4\right) \cos\left(\frac{2n-1}{2}\pi x\right) dx = \frac{(-1)^{n+1} 96 \left(\pi^2 (2n-1)^2 - 8\right)}{(2n-1)^5 \pi^5}$$

(d) [7 points] Prove that the solution $v(x,t)$ is unique. Recall that $v(x,t)$ satisfies

$$\begin{aligned} v_t &= v_{xx}, \qquad 0 < x < 1, \qquad t > 0 \\ \frac{\partial v}{\partial x}(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0 \\ v(x,0) &= f(x) \qquad 0 < x < 1. \end{aligned}$$

(e) [3 points] Assume $u_0 = 1$ and show that

$$\left|\frac{v_2(x,t)}{v_1(x,t)}\right| \leq \frac{1}{81}\frac{9\pi^2 - 8}{\pi^2 - 8} e^{-2}, \qquad t \geq 1/\pi^2.$$

(f) [3 points] Assume $u_0 = 1$ and $b > 0$. Sketch spatial (in x) profiles for $u(x,t)$ at $t = 0$, $t \to \infty$ and one intermediate spatial temperature profile $u(x, t_0)$, for $0 < x < 1$. In (e) you showed that the second term was small compared to the first, so (without proof) write down the first term approximation

$$u(x,t) \approx u_E(x) + A_1 e^{-\pi^2 t/4} \cos\left(\frac{\pi x}{2}\right)$$

which is expected to be good for $t \geq 1/\pi^2$. Write A_1 explicitly, and comment on the physical significance of its sign.