Test 2

18.303 Linear Partial Differential Equations

Matthew J. Hancock

Nov 16, 2006

1 Rules

You may only use pencils, pens, erasers, and straight edges. No calculators, notes, books or other aides are permitted. Scrap paper will be provided.

Be sure to show a few key intermediate steps and make statements in words when deriving results - answers only will not get full marks.

2 Given

The Jacobian determinant of the change of variable $(r, s) \to (x, t)$ is

$$\frac{\partial (x,t)}{\partial (r,s)} = \det \begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = x_r t_s - x_s t_r = \frac{\partial x}{\partial r}\frac{\partial t}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial r}$$

3 Questions

Total points: 53

4 Question 1

[20 points total]

Suppose you shake the end of a rope of dimensionless length 1 at a certain frequency ω. The opposite end of the rope is fixed to a wall. We aim to find the special frequencies ω at which certain points along the rope remain fixed in mid air. We negelect gravity and friction and model the waves on the rope using the 1D wave equation:

$$u_{tt} = u_{xx}, \qquad 0 < x < 1, \qquad t > 0 \tag{1}$$

where u is the displacement of the rope away from it's rest state. The end at the wall (at $x = 0$) is fixed:

$$u(0,t) = 0, \qquad t > 0. \tag{2}$$

You shake the other end (at $x = 1$) sinusoidally, with frequency ω, and give it the displacement

$$u(1,t) = \sin \omega t, \qquad t > 0. \tag{3}$$

We assume the rope has zero initial position and velocity

$$u(x,0) = 0, \qquad 0 < x < 1, \tag{4}$$

$$u_t(x,0) = 0, \qquad 0 < x < 1. \tag{5}$$

(a) [10 points] Find a solution of the form

$$u_S(x,t) = X(x) \sin \omega t \tag{6}$$

that satisfies the PDE (1) and the BCs (2) and (3). (Don't worry about the ICs yet.) Where is the rope stationary (i.e. $u_s(x,t) = 0$)? For what values of ω is your solution invalid?

Solution: From (6), we have

$$u_{Sxx} = X''(x) \sin \omega t$$

$$u_{Stt} = -\omega^2 X(x) \sin \omega t$$

Thus the PDE (1) implies

$$X'' + \omega^2 X = 0$$

Solving gives

$$X(x) = a \cos(\omega x) + b \sin(\omega x) \tag{7}$$

Substituting (1) into the BC (2) gives

$$0 = X(0)\sin\omega t$$

and hence $X(0) = 0$. Thus (7) becomes

$$0 = X(0) = a$$

and hence

$$X(x) = b\sin\omega x$$

BC (3) is satisfied if

$$1 = X(1) = b\sin\omega$$

Thus $b = 1/\sin\omega$ and

$$u_S(x,t) = \frac{\sin\omega x}{\sin\omega}\sin\omega t$$

Rope stationary at $\omega x = n\pi$,

$$x = \frac{n\pi}{\omega}, \qquad n = 1, 2, 3...$$

such that $n\pi/\omega < 1$. Solution invalid when $\omega = n\pi$ for some $n = 1, 2, 3, ...$

(b) [10 points] Use $u_S(x,t)$ from part (a) to find the full solution $u(x,t)$ to the PDE (1), the BCs (2) and (3), and the ICs (4) and (5). Hint: use $u_S(x,t)$ and $u(x,t)$ to find a wave problem for some quantity $v(x,t)$ that has Type I BCs (i.e. zero displacement) at $x = 0$, 1. Then write down D'Alembert's solution (without derivation) to satisfy the PDE and initial conditions for this problem (don't evalutate the integral in D'Alembert's solution). Then adjust D'Alembert's solution to handle the BCs at $x = 0, 1$. You don't need to evaluate the integral.

Solution: Let $v = u - u_S$. Then

$$v_{tt} = v_{xx}, \qquad 0 < x < 1, \qquad t > 0$$

$$v(0,t) = 0 = v(1,t), \qquad t > 0$$

$$\begin{aligned} v(x,0) &= u(x,0) - u_S(x,0) = 0 \\ v_t(x,0) &= u_t(x,0) - u_{St}(x,0) = 0 - \frac{\omega}{\sin\omega}\sin\omega x \end{aligned}$$

D'Alembert's solution for the infinite string is

$$v(x,t) = -\frac{\omega}{2\sin\omega}\int_{x-t}^{x+t}\sin(\omega s)\,ds$$

To satisfy the BCs, we must extend $\sin(\omega s)$ to an odd periodic function in s. Since $\sin(\omega s)$ is already odd, we merely need it's $2-$periodic extension,

$$\hat{f}(s) = \sin(\omega(s\text{mod}2))$$

where we assume $(-s)\,\text{mod}2 = -(s\text{mod}2)$, so that

$$v(x,t) = -\frac{\omega}{2\sin\omega}\int_{x-t}^{x+t}\hat{f}(s)\,ds$$

Thus

$$\begin{aligned} u(x,t) &= v(x,t) + u_S(x,t) \\ &= -\frac{\omega}{2\sin\omega}\int_{x-t}^{x+t}\sin(\omega(s\text{mod}2))\,ds \\ &\quad +\frac{\sin\omega x}{\sin\omega}\sin\omega t \end{aligned}$$

5 Question 2

[30 points total]

Consider the following quasi-linear PDE,

$$\frac{\partial u}{\partial t} + (1+2u)\frac{\partial u}{\partial x} = -u; \qquad u(x,0) = f(x) \tag{8}$$

where the initial condition is

$$f(x) = \begin{cases} 1, & |x| > 1 \\ 2-|x|, & |x| \le 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 2+x, & -1 \le x \le 0 \\ 2-x, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$

(a) [8 points] Find the parametric solution, using r as your parameter along a characteristic and s to label the characteristic (i.e. the initial value of x). First write down the relevant ODEs for $\partial t/\partial r$, $\partial x/\partial r$, $\partial u/\partial r$. Take the initial conditions $t = 0$ and $x = s$ at $r = 0$. Using the initial condition in (8), write down the IC for u at $r = 0$, in terms of s. Solve for t, u and x (in that order!) as functions of r, s. When integrating for x, be careful: u depends on r!

Solution: The parametric ODEs are

$$\frac{\partial t}{\partial r} = 1, \qquad \frac{\partial x}{\partial r} = 1+2u, \qquad \frac{\partial u}{\partial r} = -u.$$

The ICs are

$$t\left(0;s\right)=0, \qquad x\left(0;s\right)=s, \qquad u\left(0;s\right)=f\left(x\left(0;s\right)\right)=f\left(s\right) \tag{9}$$

Solving for t and u gives

$$t=r+c_1, \qquad \ln u=-r+c_2$$

where $c_{1,2}$ are constants of integration. Thus

$$u=c_3e^{-r}$$

Imposing the ICs (9) gives

$$t=r, \qquad u=f\left(s\right)e^{-r}=f\left(s\right)e^{-t}$$

Substituting u into the equation for x gives

$$\frac{\partial x}{\partial r}=1+2f\left(s\right)e^{-r}$$

and integrating yields

$$x=r-2f\left(s\right)e^{-r}+c_4$$

Imposing the IC (9) gives

$$\begin{aligned} x &= r-2f\left(s\right)\left(e^{-r}-1\right)+s \\ &= t-2f\left(s\right)\left(e^{-t}-1\right)+s \end{aligned}$$

(b) [8 points] At what time t_{sh} and position x_{sh} does your parametric solution break down? Hint: you might need to consider negative values of $f'\left(s\right)$.

Solution: The solution breaks down when the Jacobian is zero:

$$\begin{aligned} J &= \det\begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix}=\frac{\partial x}{\partial r}\frac{\partial t}{\partial s}-\frac{\partial x}{\partial s}\frac{\partial t}{\partial r}=-\frac{\partial x}{\partial s} \\ &= -\left(-2f'\left(s\right)\left(e^{-r}-1\right)+1\right) \\ &= 0 \end{aligned}$$

Since $r=t$, we solve for t to obtain

$$t_{sh}=-\ln\left(1+\frac{1}{2f'\left(s\right)}\right)$$

Note that $f'(s)$ is either 0, yielding infinite t_{sh}, 1, yielding negative t_{sh}, and -1, yielding

$$t_{sh} = -\ln\left(\frac{1}{2}\right) = \ln 2$$

Thus the shock time is at $\ln 2$. An s value where $f'(s) = -1$ is $s = 1$, so that the shock location is

$$\begin{aligned} x_{sh} &= t_{sh} - 2f(1)\left(e^{-t_{sh}} - 1\right) + 1 \\ &= \ln 2 - 2\left(\frac{1}{2} - 1\right) + 1 = \ln 2 + 2 \end{aligned}$$

(c) [4 points] Write down x in terms of t, s and $f(s)$. For each of $s = -1$, 0, 1, write down x as a function of t.

Solution:

$$\begin{aligned} s &= -1 : x = t - 2\left(e^{-t} - 1\right) - 1 \\ s &= 0 : x = t - 4\left(e^{-t} - 1\right) \\ s &= 1 : x = t - 2\left(e^{-t} - 1\right) + 1 \end{aligned}$$

(d) [4 points] Fill in the table below, using your result from either question (a) or (c) to obtain u and x at the $s-$values listed at time $t = \ln 2$ (note that $e^{-\ln 2} = \frac{1}{2}$, and you may use $\ln 2 = 0.7$):

$t = \ln 2$

$s =$	-1	0	1
$u =$			
$x =$			

Solution:

$t = \ln 2$

$s =$	-1	0	1
$u =$	1/2	1	1/2
$x =$	$\ln 2$	$\ln 2 + 2$	$\ln 2 + 2$

(e) [6 points] Plot the initial $f(x)$ vs. x. Then, on the SAME plot, plot $u(x,t)$ at $t = \ln 2$ by plotting the three points (x, u) from the table in part (d).