18.305 Fall 2004/05

Assignment 1: Series Solutions to ODEs

Revised and extended by M.S.Kilic using last year's solutions by P.Fok

1. **Problem 1.(5x5+5pts) Find and classify the singular points (including ∞) for the following ODE:**

(a) $xy'' + (c-x)y' - ay = 0$ **(confluent hypergeometric equation)**

(b) $x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0.$ **(hypergeometric equation)**

(c) $y'' - (x^4 - \frac{3}{16}x^{-2})y = 0.$

(d) $y'' + (x^2 + \frac{3}{16}x^{-2})y = 0.$

(e) $y'' + (\nu + \frac{1}{2} - \frac{1}{4}x^2)y = 0$, ν **a constant. (parabolic cylinder equation)**

The recurrence formula for the last equation involves three different a_n.Find a change of the dependable variable so that the recurrence formula for the transformed equation involves only two different a_n.

We recall that, in order to find the behaviour of the solution at very large values of x, it is convenient to let $t = \frac{1}{x}$, in which case we have

$$\begin{aligned}\frac{d}{dx} &= \frac{d}{d(\frac{1}{t})} = -t^2\frac{d}{dt} \\ \frac{d^2}{dx^2} &= t^2\frac{d}{dt}(t^2\frac{d}{dt}) = t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt}\end{aligned}$$

Thus a second order differential equation of the form

$$y'' + C(x)y' + D(x)y = 0$$

transforms into

$$\frac{d^2y}{dt^2} + \frac{2t - C(\frac{1}{t})}{t^2}\frac{dy}{dt} + \frac{D(\frac{1}{t})}{t^4}y = 0 \tag{1}$$

Solution:

(a) (confluent hypergeometric equation) First we put the equation into the form

$$y'' + (\frac{c}{x} - 1)y' - \frac{a}{x}y = 0$$

which is the right form the behaviour of the points.

The point $x = 0$ is not an ordinary point since the function $(\frac{c}{x} - 1)$ is not analytic at $x = 0$ (in case $c \neq 0$).Furthermore, since $x(\frac{c}{x} - 1)$ and $x^2(-\frac{a}{x})$ are both analytic, $x = 0$ is a regular singular point.$x = \infty$. Using (1), our equation transforms into

$$\frac{d^2y}{dt^2} + \frac{2t - (ct-1)}{t^2}\frac{dy}{dt} - \frac{at}{t^4}y = 0$$

and hence $x = \infty$ is an irregular singular point of rank 1.

(b) (hypergeometric equation)

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

$\Rightarrow$

$$y'' + \frac{[c-(a+b+1)x]}{x(1-x)}y' - \frac{ab}{x(1-x)}y = 0$$

$x=0$ and $x=1$ are regular singular points except some special values of the parameters a, b and c. For $x=\infty$, we use (1) to obtain

$$\frac{d^2y}{dt^2} + \frac{(2-c)t+(a+b-1)}{(t-1)t}\frac{dy}{dt} - \frac{ab}{t^2(t-1)}y = 0$$

Hence, $x=\infty$ is a regular singular point except some special values of the parameters a and b.

(c) $y'' - (x^4 - \frac{3}{16}x^{-2})y = 0.$

Clearly, $x=0$ is a regular singular point. With $t=\frac{1}{x}$, we have

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} - \frac{(t^{-4} - \frac{3}{16}t^2)}{t^4}y = 0$$

Therefore $x=\infty$ is an irregular singular point of rank 3. c.$y'' - (x^4 - \frac{3}{16}x^{-2})y = 0.$

(d) $y'' + (x^2 + \frac{3}{16}x^{-2})y = 0.$

Clearly, $x=0$ is a regular singular point. With $t=\frac{1}{x}$, we have

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + \frac{(t^{-2} + \frac{3}{16}t^2)}{t^4}y = 0$$

Therefore $x=\infty$ is an irregular singular point of rank 2.

(e) $y'' + (\nu + \frac{1}{2} - \frac{1}{4}x^2)y = 0.$

Since the function multiplying y is analytic everywhere, all the points except possibly $x=\infty$ are ordinary points.
For $x=\infty$, using (1), we obtain the transformed equation to be

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + \left(\frac{(\nu+\frac{1}{2})t^2 - \frac{1}{4}}{t^6}\right)y = 0$$

where $t=\frac{1}{x}$. Hence we see that $x=\infty$ is an irregular singular point of rank 2.

For the last part of the question, we observe that the original form of the equation has "operators of three different orders":
* D^2 : order -2
* $(\nu + \frac{1}{2})$: order 0

* $-\frac{1}{4}x^2$: order +2
This will lead to three different types of a_n terms in the recurrence formula. Indeed, if we let

$$y = \sum a_n x^n$$

in the original parabolic cylinder equation, we arrive at the recurrence formula

$$a_n n(n-1) + (\nu + \frac{1}{2})a_{n-2} - \frac{1}{4}a_{n-4} = 0$$

from which it is hard, if not impossible, to obtain an explicit formula for a_n's. We are asked to make a change of the dependable variable in order to get away form this problem. We tentatively let

$$y = uY$$

where u is a function of x. Then

$$[D^2 + (\nu + \frac{1}{2} - \frac{1}{4}x^2)]uY = u[(D + \frac{u'}{u})^2 + (\nu + \frac{1}{2} - \frac{1}{4}x^2)]Y = 0$$

Hence

$$[(D + \frac{u'}{u})^2 + (\nu + \frac{1}{2} - \frac{1}{4}x^2)]Y = 0$$

We look more closely at

$$(D + \frac{u'}{u})^2 + \nu + \frac{1}{2} - \frac{1}{4}x^2 = D^2 + \frac{u'}{u}D + D\frac{u'}{u} + (\frac{u'}{u})^2 + \nu + \frac{1}{2} - \frac{1}{4}x^2 \qquad (2)$$

Here we shall either eliminate the order -2 term which is $-\frac{1}{4}x^2$, or all terms with order 0. The former seems easier, if we let

$$(\frac{u'}{u})^2 - \frac{1}{4}x^2 = 0$$

then a solution to this will be

$$\frac{u'}{u} = -\frac{1}{2}x \ \Rightarrow u = e^{-\frac{1}{4}x^2}$$

Putting this into (2), we obtain

$$(D^2 + xD + \nu)Y = 0$$

as our transformed differential equation. This formulation will not have the problems that the older had, as we have only operators of two different orders, namely D^2 with order -2, and $(xD + \nu)$ with order 0.
Problem 2(7x5pts):Find the Maclaurin series solutions for the equations in problem 1. In what region is each of the series convergent?

(a) $xy'' + (c-x)y' - ay = 0$ (Confluent Hypergeometric Equation)

Since we showed in problem 1 that $x = 0$ is a regular singular point, we seek a solution in the form of a *Frobenius series:*

$$y(x) = \sum_{n=-\infty}^{\infty} a_n x^{n+s};\ a_n = 0 \text{ if } n < 0$$

where s is some constant to be determined later. However, since coefficient of y is not singular, one solution can still be obtained by expanding Taylor Series. Because, the indicial equation, which can be obtained by putting $y = x^s$, and considering only the lowest order operators, will have the root $s = 0$.
After this remark, we proceed as we should, by the Frobenius Series:

$$y' = \sum_{n=-\infty}^{\infty} (n+s)a_n x^{n+s-1},\ y'' = \sum_{n=-\infty}^{\infty} (n+s)(n+s-1)a_n x^{n+s-2} \tag{3}$$

Substituting this series into the original equation, we obtain

$$\sum_{n=-\infty}^{\infty} (n+s)(n+s-1)a_n x^{n+s-2} + \sum_{n=-\infty}^{\infty} c(n+s)a_n x^{n+s-1} - \sum_{n=-\infty}^{\infty} [(n+s)+a]a_n x^{n+s} = 0$$

Relabeling indices,

$$\sum_{n=-\infty}^{\infty} [(n+1+s)(n+s)a_{n+1} + [c(n+1+s) - (n+s) - a]a_n]x^{n+s} = 0$$

And the recurrence relation is

$$(n+s+1)(n+s+c)a_{n+1} - (n+s+a)a_n = 0$$

Setting $n = -1$ yields ($a_{-1} = 0$, $a_0 \neq 0$) the indicial equation

$$s(s-1+c) = 0$$

Therefore s can take two values, either $s = 0$ or $s = 1 - c$. We'll treat each case separately:
For $s = 0$,

$$a_{n+1} = \frac{n+a}{(n+1)(n+c)}a_n$$

where c cannot be a negative integer, or zero. Then,

$$\begin{aligned} a_n &= \frac{n-1+a}{n(n-1+c)}a_{n-1} \\ &= \frac{(n-1+a)(n-2+a)(1+a)}{n!(n-1+c)(n-2+c)...(1+c)c}a_0 \\ &= \frac{\Gamma(n+a)\Gamma(c)}{\Gamma(a)\Gamma(n+1)\Gamma(n+c)}a_0 \end{aligned}$$

Therefore the solution corresponding to $s = 0$ must be proportional to

$$\boxed{\mathrm{y}_1(\mathrm{x}) = \sum_{n=0}^{\infty} \frac{\Gamma(n+a)}{n!\Gamma(n+c)} \mathrm{x}^n}$$

For $s = 1 - c$,

$$a_{n+1} = \frac{n+1-c+a}{(n+2-c)(n+1)} a_n$$

or

$$a_n = \frac{n-c+a}{(n+1-c)n)} a_{n-1}$$

where $c \neq 2, 3, 4, ...$ Then we find

$$a_n = \frac{\Gamma(n+1+a-c)\Gamma(2-c)}{\Gamma(1+a-c)\Gamma(n+2-c)} a_0$$

and the second solution is proportional to

$$\boxed{y_2(x) = x^{1-c} \sum_{n=0}^{\infty} \frac{\Gamma(n+1+a-c)}{n!\Gamma(n+2-c)} x^n}$$

This will be independent from $y_1(x)$ provided $c \neq 1$. The general solution of the confluent hypergeometric equation will be a linear combination of $y_1(x)$ and $y_2(x)$.
As can be seen by applying the ratio test, both series are convergent for $|x| < \infty$.

(b) $x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$. (hypergeometric equation)

$x = 0$ is a regular singular point, and so the solution is in the form of a Frobenius series

$$y(x) = \sum_{n=-\infty}^{\infty} a_n x^{n+s}; \ a_n = 0 \text{ if } n < 0$$

and y' and y'' are the same as before.Also same remarks apply as in (a). Substituting, we obtain

$$0 = \sum_{n=-\infty}^{\infty} (n+s)(n+s-1)a_n x^{n+s-1} - \sum_{n=-\infty}^{\infty} (n+s)(n+s-1)a_n x^{n+s}$$
$$+ \sum_{n=-\infty}^{\infty} c(n+s)a_n x^{n+s-1} - \sum_{n=-\infty}^{\infty} [(a+b+1)(n+s)+ab]a_n x^{n+s}$$

Relabeling indices,

$$\sum_{n=-\infty}^{\infty} [(n+1+s)(n+s+c)a_{n+1} - [(n+s)(n+s+a+b)+ab]a_n]x^{n+s} = 0$$

By equating each term is zero, and factorizing the coefficient of a_n

$$(n+s)(n-1+s+c)a_{n+1} - (n-1+s+a)(n-1+s+b)a_{n-1} = 0$$

With $n = 0$, the indicial equation is

$$s(s+c-1) = 0$$

Therefore, either $s = 0$ or $s = 1 - c$.
For $s = 0$,

$$\begin{aligned} a_{n+1} &= \frac{(n-1+a)(n-1+b)}{n(n-1+c)} a_{n-1} \\ &= \frac{(n-1+a)(n-2+a)...a(n-1+b)(n-2+b)...b}{n!(n-1+c)(n-2+c)...c} a_0 \end{aligned}$$

This is all done assuming that c is not zero, or a negative integer. Then one of the solutions is found to be

$$\boxed{\mathrm{y}_3(\mathrm{x}) = \sum_{n=0}^{\infty} \frac{\Gamma(n+a)\Gamma(n+b)}{n!\Gamma(n+c)} \mathrm{x}^n}$$

For $s = 1 - c$,

$$\begin{aligned} a_n &= \frac{(n-c+a)(n-c+b)}{n(n+1-c)} a_{n-1} \\ &= \frac{(n-c+a)(n-1-c+a)...(1-c+a)(n-c+b)(n-1-c+b)...(1-c+b)}{n!(n+1-c)(n+c)...(2-c)} a_0 \\ &= \frac{\Gamma(n-c+a+1)\Gamma(n-c+b+1)\Gamma(2-c)}{n!\Gamma(1-c+a)\Gamma(1-c+b)\Gamma(n+2-c)} a_0 \end{aligned}$$

Here we assumed that c is not positive integer. Then we find

$$\boxed{\mathrm{y}_4(\mathrm{x}) = \mathrm{x}^{1-c} \sum_{n=0}^{\infty} \frac{\Gamma(n+a+1-c)\Gamma(n+b+1-c)}{n!\Gamma(n+2-c)} \mathrm{x}^n}$$

And the most general solution to the Hypergeometric Equation is a linear combination of $y_3(x)$ and $y_4(x)$, providing that $c \neq 0, \pm 1, \pm 2, ...$ The series are convergent for all $|x| < 1$.

(c) $y'' - (x^4 - \frac{3}{16}x^{-2})y = 0.$

Since we know that $x = 0$ is a regular singular point, we expand a Frobenius Series $y(x) = \sum_{n=-\infty}^{\infty} a_n x^{n+s}$; $a_n = 0$ if $n < 0$. Substituting y'' as given in (3) into the equation, we obtain

$$\sum_{n=-\infty}^{\infty} a_n(n+s)(n+s-1)x^{n+s-2} - \sum_{n=-\infty}^{\infty} a_n x^{n+s+4} + \sum_{n=-\infty}^{\infty} \frac{3}{16} a_n x^{n+s-2} = 0$$

$\Rightarrow$

$$\sum_{n=-\infty}^{\infty} \{a_n[(n+s)(n+s-1) + \frac{3}{16}] - a_{n-6}\} x^{n-2} = 0$$

Setting $n = 0$ gives the indicial equation $(a_0 \neq 0, a_{-6} = 0)$

$$s(s-1) + \frac{3}{16} = 0$$

which implies either $s = \frac{1}{4}$ or $s = \frac{3}{4}$.

For $s = \frac{1}{4}$, we have

$$a_n[(n+\frac{1}{4})(n-\frac{3}{4})+\frac{3}{16}] = a_{n-6}$$

i.e

$$a_n = \frac{a_{n-6}}{n(n-\frac{1}{2})}$$

Then putting $n = 6m$, we obtain

$$a_{6m} = \frac{a_{6(m-1)}}{6^2 m(m-\frac{1}{12})}$$

Therefore, we conclude

$$a_{6m} = \frac{\Gamma(\frac{11}{12})}{6^{2m} m! \Gamma(m+\frac{11}{12})} a_0$$

and the solution is proportional to

$$\boxed{y_5(x) = x^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{x^{6n}}{6^{2n} n! \Gamma(n+\frac{11}{12})}}$$

Clearly, the series is convergent for all $|x| < \infty$ (*say*, by ratio test).
For $s = \frac{3}{4}$,we have

$$a_n[(n+\frac{3}{4})(n-\frac{1}{4})+\frac{3}{16}] = a_{n-6}$$

i.e

$$a_n = \frac{a_{n-6}}{n(n+\frac{1}{2})}$$

Then putting $n = 6m$, we obtain

$$a_{6m} = \frac{a_{6(m-1)}}{6^2 m(m+\frac{1}{12})}$$

Therefore, we conclude

$$a_{6m} = \frac{\Gamma(\frac{13}{12})}{6^{2m} m! \Gamma(m+\frac{13}{12})} a_0$$

and the solution is proportional to

$$\boxed{y_6(x) = x^{\frac{3}{4}} \sum_{n=0}^{\infty} \frac{x^{6n}}{6^{2n} n! \Gamma(n+\frac{13}{12})}}$$

Clearly, the series is convergent for all $|x| < \infty$ (*say*, by ratio test).
The general solution to the differential equation will then be a linear combination of those two solutions, namely $y_5(x)$ and $y_6(x)$, for $|x| < \infty$.

(d) $y'' + (x^2 + \frac{3}{16} x^{-2})y = 0.$

Since we know that $x=0$ is a regular singular point, we expand a Frobenius Series $y(x) = \sum_{n=-\infty}^{\infty} a_n x^{n+s}$; $a_n = 0$ if $n<0$. Substituting y'' as given in (3) into the equation, we obtain

$$\sum_{n=-\infty}^{\infty} a_n(n+s)(n+s-1)x^{n+s-2} + \sum_{n=-\infty}^{\infty} a_n x^{n+s+2} + \sum_{n=-\infty}^{\infty} \frac{3}{16} a_n x^{n+s-2} = 0$$

$\Rightarrow$

$$\sum_{n=-\infty}^{\infty} \{a_n[(n+s)(n+s-1)+\frac{3}{16}] + a_{n-4}\}x^{n-2} = 0$$

Setting $n=0$ gives the indicial equation $(a_0 \neq 0, a_{-4} = 0)$

$$s(s-1)+\frac{3}{16} = 0$$

which implies either $s=\frac{1}{4}$ or $s=\frac{3}{4}$.
For $s=\frac{1}{4}$, we have

$$a_n[(n+\frac{1}{4})(n-\frac{3}{4})+\frac{3}{16}] = -a_{n-4}$$

i.e

$$a_n = -\frac{a_{n-4}}{n(n-\frac{1}{2})}$$

Then putting $n=4m$, we obtain

$$a_{4m} = (-1)^m \frac{a_{4(m-1)}}{4^2 m(m-\frac{1}{8})}$$

Therefore, we conclude

$$a_{4m} = (-1)^m \frac{\Gamma(\frac{7}{8})}{4^{2m} m! \Gamma(m+\frac{7}{8})} a_0$$

and the solution is proportional to

$$\boxed{y_7(x) = x^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{4^{2n} n! \Gamma(n+\frac{7}{8})}}$$

Clearly, the series is convergent for all $|x| < \infty$ (*say*, by ratio test).
For $s=\frac{3}{4}$, we have

$$a_n[(n+\frac{3}{4})(n-\frac{1}{4})+\frac{3}{16}] = -a_{n-4}$$

i.e

$$a_n = -\frac{a_{n-4}}{n(n+\frac{1}{2})}$$

Then putting $n=4m$, we obtain

$$a_{4m} = (-1)^m \frac{a_{4(m-1)}}{4^2 m(m+\frac{1}{8})}$$

Therefore, we conclude

$$a_{4m} = (-1)^m \frac{\Gamma(\frac{9}{8})}{4^{2m} m! \Gamma(m+\frac{9}{8})} a_0$$

and the solution is proportional to

$$\boxed{\mathrm{y}_8(\mathrm{x})=\mathrm{x}^{\frac{1}{4}}\sum_{n=0}^{\infty}(\text{-}1)^n\frac{x^{4n}}{4^{2n}n!\Gamma(n+\frac{9}{8})}}$$

Clearly, the series is convergent for all $|x| < \infty$ (*say*, by ratio test).
The general solution of the differential equation then is a linear combination of the two solutions found above.

(e) $y'' + (\nu + \frac{1}{2} - \frac{1}{4}x^2)y = 0$, ν **a constant. (parabolic cylinder equation)**

We remember from problem(1.*e*) that our transformed equation was

$$Y'' - xY + \nu Y = 0$$

where $y = e^{-\frac{1}{4}x^2}Y$. Since now $x = 0$ is an ordinary point, we simply expand the Taylor Series $Y(x) = \sum_{n=-\infty}^{\infty} a_n x^n$. Then $Y''(x) = \sum_{n=-\infty}^{\infty} n(n-1)a_n x^{n-2}$. Substituting those into the equation, we obtain

$$\sum_{n=-\infty}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=-\infty}^{\infty} n a_n x^n + \sum_{n=-\infty}^{\infty} \nu a_n x^n = 0$$

or

$$\sum_{n=-\infty}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=-\infty}^{\infty} (n-\nu)a_n x^n = 0$$

which gives

$$(n+2)(n+1)a_{n+2} - (n-\nu)a_n = 0$$

Then

$$a_n = \frac{(n-2-\nu)}{n(n-1)}a_{n-2} \tag{4}$$

Letting $n = 2m$,

$$a_{2m} = \frac{2(m-1-\frac{\nu}{2})}{2^2 m(m-\frac{1}{2})}a_{2(m-1)}$$

Then

$$a_{2m} = \frac{2(m-1-\frac{\nu}{2})}{2^2 m(m-\frac{1}{2})}a_{2(m-1)}$$

Then

$$a_{2m} = \frac{\Gamma(m-\frac{\nu}{2})\Gamma(\frac{1}{2})}{2^m m!\Gamma(m+\frac{1}{2})\Gamma(-\frac{\nu}{2})}a_0$$

Therefore one of the solutions is proportional to

$$\boxed{\mathrm{y}_9(\mathrm{x})=\mathrm{e}^{-\frac{1}{4}x^2}\sum_{n=0}^{\infty}\frac{\Gamma(n-\frac{\nu}{2})}{2^n n!\Gamma(n+\frac{1}{2})}\mathrm{x}^{2n}}$$

The series is convergent for all $|x| < \infty$. To find the other solution, we let $n = 2m+1$ in (4), to obtain

$$a_{2m+1} = \frac{2m-1-\nu}{2m(2m+1)} a_{2(m-1)+1} = \frac{(m - \frac{\nu+1}{2})}{2m(m+\frac{1}{2})} a_{2(m-1)+1}$$

Then

$$a_{2m+1} = \frac{\Gamma(m - \frac{\nu-1}{2})\Gamma(\frac{1}{2})}{2^m m! \Gamma(m+\frac{3}{2})\Gamma(-\frac{1+\nu}{2})} a_1$$

Therefore one of the solutions is proportional to

$$\boxed{\mathrm{y}_{10}(\mathrm{x}) = \mathrm{e}^{-\frac{1}{4}x^2} \sum_{n=0}^{\infty} \frac{\Gamma(n - \frac{\nu-1}{2})}{2^n n! \Gamma(n+\frac{3}{2})} \mathrm{x}^{2n+1}}$$

The series is convergent for all $|x| < \infty$.
The general solution can be written as a linear combination of $y_9(x)$ and $y_{10}(x)$.

Problem 3(25pts): Calculate the Quantity

$$\lim_{p \to n} \frac{J_p(x) - \cos \pi p J_{-p}(x)}{\sin \pi p}$$

From now on, let us call this quantity $Q_n(x)$. Since

$$J_p(x) = e^{p \log\left(\frac{x}{2}\right)} \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m! \Gamma(m+p+1)}$$

$$\frac{\partial J_p(x)}{\partial p} = \log\left(\frac{x}{2}\right) J_p(x) - \left(\frac{x}{2}\right)^p \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m} \Gamma'(m+p+1)}{m! \Gamma^2(m+p+1)}$$

And of course

$$\frac{\partial J_{-p}(x)}{\partial p} = -\log\left(\frac{x}{2}\right) J_{-p}(x) + \left(\frac{x}{2}\right)^{-p} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m} \Gamma'(m-p+1)}{m! \Gamma^2(m-p+1)}$$

The numerator and denominator of $Q_n(x)$ both tend to zero as $p \to n$, so we must use L'Hopital's Rule to obtain the limit by differentiating the top and bottom with respect to p.

$$Q_n(x) = \lim_{p \to n} \frac{\frac{\partial J_p(x)}{\partial p} - \left(-\pi \sin \pi p J_{-p}(x) + \cos \pi p \frac{\partial J_{-p}(x)}{\partial p}\right)}{\pi \cos \pi p}$$

The middle term tends to zero in the limit, and since $\cos \pi p = (-1)^p$,

$$(-1)^n Q_n(x) = \frac{1}{\pi} \left(\frac{\partial J_n(x)}{\partial n} - (-1)^n \frac{\partial J_{-n}(x)}{\partial n} \right)$$

$$\therefore (-1)^n Q_n(x) = \frac{1}{\pi}\left(\log\left(\frac{x}{2}\right) J_n(x) + (-1)^n \log\left(\frac{x}{2}\right) J_{-n}(x)\right)$$
$$- \frac{1}{\pi}\left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m} \Gamma'(m+n+1)}{m!\Gamma^2(m+n+1)}$$
$$- \frac{1}{\pi}(-1)^n \left(\frac{x}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m} \Gamma'(m-n+1)}{m!\Gamma^2(m-n+1)}$$

$$\therefore (-1)^n Q_n(x) = \frac{2}{\pi}\log\left(\frac{x}{2}\right) J_n(x) - \frac{\left(\frac{x}{2}\right)^n}{\pi} \sum_{m=0}^{\infty} \frac{\left(-\frac{1}{4}x^2\right)^m \Gamma'(m+n+1)}{m!\Gamma^2(m+n+1)}$$
$$- \frac{1}{\pi}\sum_{m=0}^{n-1} \frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m-n} \Gamma'(m-n+1)}{m!\Gamma^2(m-n+1)} \tag{5}$$
$$- \frac{1}{\pi}\sum_{m=n}^{\infty} \frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m-n} \Gamma'(m-n+1)}{m!\Gamma^2(m-n+1)}$$

where I have used the fact that $J_n(x) = (-1)^n J_n(x)$ to obtain the first log term above. I have also split up the last infinite series into two parts - one involving a sum from zero to $n-1$, and the other from n to ∞. The reason for this is because for the third part of $(-1)^n Q_n(x)$ above, the arguments of the gamma functions are negative integers and require more analysis. We now consider the three sums separately. If we define $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ (ψ is then the *digamma function*), then the first of the sums

$$-\frac{\left(\frac{x}{2}\right)^n}{\pi} \sum_{m=0}^{\infty} \frac{\left(-\frac{1}{4}x^2\right)^m \Gamma'(m+n+1)}{m!\Gamma^2(m+n+1)} = -\frac{\left(\frac{x}{2}\right)^n}{\pi} \sum_{m=0}^{\infty} \frac{\left(-\frac{1}{4}x^2\right)^m \psi(m+n+1)}{m!(m+n)!}$$

The properties of the Digamma function $\psi(x)$ are not used anywhere in this question - if you're not comfortable with it, just assume it's a shorthand notation for $\frac{\Gamma'(x)}{\Gamma(x)}$.

The last sum in (5) can be simplified by relabelling the summing index. If we let $s = m - n$, then

$$-\frac{1}{\pi}\sum_{m=n}^{\infty} \frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m-n} \Gamma'(m-n+1)}{m!\Gamma^2(m-n+1)} = -\frac{\left(\frac{x}{2}\right)^n}{\pi} \sum_{s=0}^{\infty} \frac{\left(-\frac{1}{4}x^2\right)^s \psi(s+1)}{s!(n+s)!}$$

We finally come to the third sum in (5):

$$-\frac{1}{\pi}\sum_{m=0}^{n-1} \frac{(-1)^{m+n}\left(\frac{x}{2}\right)^{2m-n} \Gamma'(m-n+1)}{m!\Gamma^2(m-n+1)}$$

Note that the arguments in the gamma functions are all negative integers. We have to be careful here, because $\Gamma(x)$ becomes infinite for negative integers x, and so $\Gamma'(x)$ is certainly not well defined for negative integers either. If you remember, the ratio of the Gamma function's derivative to the function squared came originally from differentiating the *reciprical* of the function, when we calculated $\frac{\partial J_{-p}(x)}{\partial p}$. What we require then is

$$\frac{d}{dx}\frac{1}{\Gamma(x)}\bigg|_{m-n+1} \tag{6}$$

Using the "reflection formula" (should be given in Bender and Orszag) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we have that

$$\frac{d}{dx}\frac{1}{\Gamma(x)}\bigg|_{m-n+1} = \frac{d}{dx}\Gamma(1-x)\frac{\sin \pi x}{\pi}\bigg|_{m-n+1}$$

which can be calculated in a straight-forward way. We find that

$$-\frac{\Gamma'(m-n+1)}{\Gamma^2(m-n+1)} = -\Gamma(n-m)(-1)^{n-m}$$

Putting all four parts of $Q_n(x)$ together, we finally have

$$\begin{aligned} Q_n(x)(-1)^n =& \frac{2}{\pi}\log\left(\frac{x}{2}\right)J_n(x) \\ &- \frac{\left(\frac{x}{2}\right)^n}{\pi}\sum_{m=0}^{\infty}\frac{\left(-\frac{1}{4}x^2\right)^m}{m!(m+n)!}(\psi(m+n+1)+\psi(m+1)) \\ &- \frac{\left(\frac{x}{2}\right)^{-n}}{\pi}\sum_{m=0}^{n-1}\frac{\left(\frac{x^2}{4}\right)(n-m-1)!}{m!} \end{aligned}$$

The quantity $Q_n(x)(-1)^n$ is actually a Bessel function of the second kind, sometimes also called a Neumann function - denoted usually by $Y_n(x)$. It is a solution to Bessel's Equation, but is linearly *independent* to both $J_n(x)$ and $J_{-n}(x)$

Problem 4(10pts): Let $y_1(x) = (1+x)$ and $y_2(x) = x^2$ be the two independent solutions of equation $y'' + c(x)y' + d(x)y = 0$. Find $c(x)$ and d(x) and show that both of them have a simple pole at $x = 0$. What do you learn from this example.

Plugging in y_1 and y_2 in, we obtain

$$\begin{aligned} c(x) + (1+x)d(x) &= 0 \\ 2 + 2xc(x) + x^2 d(x) &= 0 \end{aligned}$$

Solving this set of equations, we find

$$c(x) = -\frac{2(x+1)}{x(x+2)},\ d(x) = \frac{2}{x(x+2)}$$

which, obviously, have simple poles at $x = 0$.
Therefore the differential equation in question has a (regular) singular point at $x = 0$. Yet, it has two independent solutions which are analytic everywhere. The moral of this example is that, that a differential equation has singularity at a point does not guarantee that it will have singular solutions around that point.