Course 18.312: Algebraic Combinatorics

Homework # 7

Due Wednesday April 1, 2009

You may discuss the homework with other students in the class, but please write the names of your collaborators at the top of your assignment. Please be advised that you should not just obtain the solution from another source. Please explain your reasoning to receive full credit, even for computational questions.

1) An **involution** is a permutation π with the property that π^2 is the identity.

   (15 points) Give a formula for the number of involutions of $\{1, 2, \ldots, n\}$ in terms of f^λ's.

2) (10 points) Show that

$$A(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}.$$

   (5 points) Give a quadratic equation that $A(x)$ satisfies.

   (Bonus 5 points) Give a cubic equation that $B(x) = \sum_{n=0}^{\infty} \binom{3n}{n} x^n$ satisfies.

3) (5 points) Let $\ell - i - j = 2m$ and

$$b_{ij}(\ell) = \frac{\ell!}{2^m i! j! m!}.$$

   Show that the $b_{ij}(\ell)$'s satisfy the recurrence

$$b_{ij}(\ell+1) = b_{i,j-1}(\ell) + (i+1)b_{i+1,j}(\ell) + b_{i-1,j}(\ell).$$

   Let λ be a partition of 9. Let $c(\lambda)$ be the number of ways to **delete** a square, **insert** a square, **delete** a square, and finally **insert** a square to return to partition λ. (Note: at each step when we delete or add a square, we assume that the resulting shape is still a partition.)

   (10 points) What is $\sum_{\lambda \vdash 9} c(\lambda)$?

4) In this problem, we give a proof of the hook formula for f^λ, the number of SYT of shape λ, using a sequence of algebraic identities. Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ be a partition of n with at most k parts, and $\ell_i = \lambda_i + k - i$ for each $1 \leq i \leq k$.

(10 points) Using the combinatorial interpretation of f^λ as the number of SYT's of shape λ, show the recursion

$$f^{[\lambda_1,\ldots,\lambda_k]} = \sum_{r=1}^{k} f^{[\lambda_1,\ldots,\lambda_r-1,\ldots,\lambda_k]}$$

where we define $f^{[\lambda_1,\ldots,\lambda_r-1,\ldots,\lambda_k]} = 0$ if sequence $[\lambda_1, \ldots, \lambda_r - 1, \ldots, \lambda_k] = 0$ is not weakly decreasing.

Let $\Delta(x_1, \ldots, x_k) = \prod_{1\leq i<j\leq k}(x_i - x_j)$ (the Vandermonde determinant)

(10 points) Show that

$$\sum_{r=1}^{k} x_r \Delta(x_1, \ldots, x_r + t, \ldots, x_k) = (x_1 + x_2 + \cdots + x_k + \binom{k}{2} t) \cdot \Delta(x_1, \ldots, x_k). \quad (1)$$

Hint 1: Show that both the left-hand and right-hand sides are antisymmetric; i.e. if we switch x_i and x_j each expression becomes its negative.

Hint 2: Evaluate each side at the values $x_i = k - i$ and $t = 1$.

(10 points) Letting $\ell_i = \lambda_i + k - i$ and $|\lambda| = n$, use (1) to show

$$n \cdot \Delta(\ell_1, \ldots, \ell_k) = \sum_{r=1}^{k} \ell_r \cdot \Delta(\ell_1, \ldots, \ell_r - 1, \ldots, \ell_k). \quad (2)$$

Let $F(\ell_1, \ldots, \ell_k) = \frac{n! \cdot \Delta(\ell_1,\ldots,\ell_k)}{\ell_1! \ell_2! \cdots \ell_k!}$.

(10 points) Show that identity (2) is equivalent to

$$F(\ell_1, \ldots, \ell_k) = \sum_{r=1}^{k} F(\ell_1, \ldots, \ell_r - 1, \ldots, \ell_k).$$

(10 points) Show that

$$F(\ell_1, \ldots, \ell_k) = \frac{n!}{\prod_{c\in\lambda h_\lambda(c)}}$$

(5 points) Conclude that $f^\lambda = \frac{n!}{\prod_{c\in\lambda h_\lambda(c)}}$.

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