Course 18.312: Algebraic Combinatorics

Homework # 8

Due Wednesday April 8, 2009

You may discuss the homework with other students in the class, but please write the names of your collaborators at the top of your assignment. Please be advised that you should not just obtain the solution from another source. Please explain your reasoning to receive full credit, even for computational questions.

1) (10 points) Expand $h_{3,1}$ in terms of the e_λ's.

   (10 points) Expand $e_{2,2}$ in terms of the p_λ's.

   (10 points) Compute $s_{3,1,1}(x_1, x_2, x_3)$ as a polynomial in terms of $\{x_1, x_2, x_3\}$.

   Hint: This computation can be done three different ways using definitions and methods from class.

   (**Bonus 5 points**) Compute $s_{3,1,1}(x_1, x_2, x_3)$ via a second method.

2) **For this problem, you are encouraged to do computer experimentation.**

   (5 points) Compute $s_{2,1}(x_1, x_2, x_3)$

   (5 points) Compute $s_{3,2,1}(x_1, x_2, x_3, x_4)$

   (**Bonus 5 points**) Do you have a conjecture for $s_{k,k-1,\ldots,3,2,1}(x_1, x_2, \ldots, x_{k+1})$? Prove it.

   (5 points) Compute $f(x_1, x_2, x_3) = e_1(x_1, x_2, x_3)s_{2,1}(x_1, x_2, x_3)$ as a polynomial in terms of $\{x_1, x_2, x_3\}$.

   (10 points) Write $f(x_1, x_2, x_3)$ as a $\mathbb{Z}$-linear combination of $s_4(x_1, x_2, x_3)$, $s_{31}(x_1, x_2, x_3)$, $s_{22}(x_1, x_2, x_3)$, $s_{211}(x_1, x_2, x_3)$, and $s_{1111}(x_1, x_2, x_3)$.

   Hint:

$$\begin{aligned} s_{3,1}(x_1, x_2, x_3) &= 2x_1^2x_2x_3 + 2x_1x_2^2x_3 + 2x_1x_2x_3^2 + x_2^3x_3 + x_2^2x_3^2 + x_1^3x_2 \\ &+ x_2x_3^3 + x_1^2x_3^2 + x_1^2x_2^2 + x_1^3x_3 + x_1x_3^3 + x_1x_2^3 \end{aligned}$$

3) Ten balls are stacked in a triangular array with 1 atop 2 atop 3 atop 4. (Think of billiards.) The triangular array is free to rotate.

(10 points) Find the generating function for the number of inequivalent colorings $r_1, r_2, \dots, r_{10}$. (You do not need to simplify your answer.)

(5 points) How many inequivalent colorings have four periwinkle balls, three teal balls, and three celadon balls?

(5 points) How many inequivalent colorings have four burgundy balls, four fuchsia balls, and two taupe balls?

4) For any finite group G of permutations of an ℓ-element set X, let $f(n)$ be the number of inequivalent (under the action of G) colorings of X with n colors.

(15 points) Find $\lim_{n\to\infty} f(n)/n^{\ell}$. Interpret your answer as saying that "most" colorings of X are asymmetric (have no symmetries).

5) Let $\{N_1, N_2, N_3, \dots\}$ be a sequence of positive integers with the property that

$$\exp\left(\sum_{k=1}^{\infty} \frac{N_k}{k} T^k\right) = \frac{(1-a_1T)(1-a_2T)\cdots(1-a_mT)}{(1-b_1T)(1-b_2T)\cdots(1-b_nT)}.$$

(10 points) Show that $N_k = b_1^k + b_2^k + \cdots + b_n^k - a_1^k - a_2^k - \cdots - a_m^k$.

Bonus) The names of one hundred prisoners are placed in one hundred wooden boxes, one name to a box, and the boxes are lined up on a table in a room. One by one, the prisoners are led into the room; they may look into up to fifty of the boxes to try to find their own name, but must leave the room exactly as it was. They are permitted no further communication after leaving the room. The prisoners have a chance to plot a strategy in advance, and they are going to need it, because unless *they all find their own names* they will all be executed.

(**Bonus 10 points**) There is a strategy that has a probability of success exceeding thirty percent - find it.

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