Course 18.312: Algebraic Combinatorics

Lecture Notes # 1 Addendum by Gregg Musiker

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1 Recurrence Relations and Generating Functions

Given an infinite sequence of numbers, a **generating function** is a compact way of expressing this data. We begin with the notion of **ordinary** generating functions. To illustrate this definition, we start with the example of the Fibonacci numbers.

$$\{F_n\}_{n=0}^{\infty} = \{F_0, F_1, F_2, F_3, \ldots\}$$

defined by $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

We define

$$\begin{aligned} F(X) &:= F_0 + F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \ldots \\ &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \ldots \ . \end{aligned}$$

In other words, $F(X)$ is the **formal power series** $\sum_{k=0}^{\infty} F_k x^k$.

Remark. This is called a "formal" power series because we will consider x to be an indeterminate variable rather than a specific real number.

In general, given a sequence of numbers $\{a_i\}_{i=0}^{\infty} = \{a_0, a_1, a_2, a_3, \ldots\}$, the associated formal power series is

$$A(X) := \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \ .$$

We will shortly write down $F(X)$ in a compact form, but we begin with an easier example that you have already seen.

Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. For example,

$$\left\{\binom{8}{k}\right\}_{k=0}^{8} = \{1, 8, 28, 56, 70, 56, 28, 8, 1\}.$$

In fact if $k > 8$, $\binom{8}{k}$ (e.g. $\binom{8}{9}$) equals zero. Thus we can consider the entire infinite sequence as

$$\left\{\binom{8}{k}\right\}_{k=0}^{\infty} = \{1, 8, 28, 56, 70, 56, 28, 8, 1, 0, 0, 0, \ldots\},$$

and then the associated formal power series

$$1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8 + 0x^9 + 0x^{10} + \ldots$$

can be written compactly as $(1+x)^8$.

Generalizing this to any positive integer n, $\left\{\binom{n}{k}\right\}_{k=0}^{\infty}$ has associated power series $(1+x)^n$, since $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$ by the **Binomial Theorem**.

This illustrates that from a formal power series, we can recover a sequence of numbers. We call these numbers the **coefficients** of the formal power series. For example, we say that $\binom{n}{k}$ is the coefficient of x^k in $(1+x)^n$. This is sometimes written as $(1+x)^n\Big|_{x^k} = \binom{n}{k}$ or $[x^k](1+x)^n = \binom{n}{k}$.

1.1 More complicated formal power series

We now want to write a similar expression for $F(X) = \sum_{k=0}^{\infty} F_k x^k$, where $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$ and $F_0 = F_1 = 1$.

Notice that

$$\sum_{k=0}^{\infty} a_k x^k \pm \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} (a_k \pm b_k) x^k.$$

As a consequence, $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$ implies

$$\begin{aligned} F(X) &= 1 + F_1 x + \sum_{k=2}^{\infty} F_k x^k \\ &= 1 + F_1 x + \sum_{k=2}^{\infty} (F_{k-1} + F_{k-2}) x^k \\ &= 1 + F_1 x + \sum_{k=2}^{\infty} F_{k-1} x^k + \sum_{k=2}^{\infty} F_{k-2} x^k \\ &= 1 + F_1 x + \Big(xF(X) - F_0 x\Big) + x^2 F(X). \end{aligned}$$

Thus

$$F(X)(1 - x - x^2) = 1 + (F_1 - F_0)x = 1 + 0x$$

and we obtain the rational expression

$$F(X) = \frac{1}{1 - x - x^2}.$$

If we look at the Taylor series for this rational function, we indeed obtain coefficients that are the Fibonacci numbers. Generating Functions are also helpful for obtaining closed formulas or asymptotic formulas.

If we use partial fraction decomposition, we see that

$$F(X) = \frac{A}{1 - \lambda_1 x} + \frac{B}{1 - \lambda_2 x}.$$

We know that $(1 - \lambda_1 x)(1 - \lambda_2 x) = 1 - x - x^2$ so

$$\begin{aligned} \lambda_1 \lambda_2 &= -1 \quad \text{and} \\ -\lambda_1 - \lambda_2 &= -1 \end{aligned}$$

Thus $\{\lambda_1, \lambda_2\} = \{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$.

Exercise 1: Solve for A and B and use this to obtain a closed form expression for F_k.

Notice that as a consequence we can compute that $\{F_{k+1}/F_k\} = \{1/1, 2/1, 3/2, 5/3, 8/5, 13/8, \ldots\}$ converges to $\frac{1+\sqrt{5}}{2}$ since $\left(\frac{1-\sqrt{5}}{2}\right)^k \to 0$ as $k \to \infty$, so

$$\frac{A\lambda_1^{k+1} + B\lambda_2^{k+1}}{A\lambda_1^k + B\lambda_2^k} \to \frac{A\lambda_1^{k+1}}{A\lambda_1^k} = \lambda_1.$$

1.2 A Combinatorial Interpretation of the Fibonacci Numbers

Given a sequence of integers $S = \{s_0, s_1, s_2, \ldots\}$, a **combinatorial interpretation** of S is a family $\mathcal{F}$ of objects (of various sizes) such that the number of objects in $\mathcal{F}$ of size k is exactly counted by s_k.

For example, a combinatorial interpretation of $\binom{n}{k}$ is as the number of subsets of an $\{1, 2, \ldots, n\}$ of size k.

A **domino tiling** of a rectangular region R is a covering of R by horizontal (1-by-2) domino tiles and vertical (2-by-1) domino tiles such that every square of R is covered by exactly one domino.

For example, if we let R be a 2-by-2 grid, then there are two possible domino tilings. Either both tiles are vertical or both are horizontal. If we let R be a 2-by-3 grid, then there are three possible domino tilings, and a 2-by-4 grid would have five such domino tilings.

Proposition. The number of domino tilings of a 2-by-n grid is counted by the nth Fibonacci number, F_n for $n \geq 1$.

Proof. Let DT_n denote the number of domino tilings of the 2-by-n grid. We first check the initial conditions. There is one way to tile the 2-by-1 grid, and there are two ways to tile the 2-by-2 grid. Thus $DT_1 = 1 = F_1$ and $DT_2 = 2 = F_2$. (Recall that $F_0 = 1$, but we do not use this quantity in this combinatorial interpretation.)

Domino tilings of the 2-by-n grid either look like a domino tiling of the 2-by-$(n-1)$ grid with a vertical tile tacked onto the end, or a domino tiling of the 2-by-$(n-2)$ grid with two horizontal tiles tacked onto the end. Consequently, $DT_n = DT_{n-1} + DT_{n-2}$, the same recurrence as the F_n's.

Exercise 2: Show that this combinatorial interpretation can be rephrased as the statement

$$F_n = \text{ The number of subsets } S \text{ of } \{a_1, a_2, \ldots, a_{n-1}\}$$

such that a_i and a_{i+1} are not both contained in S.

1.3 Convolution Product Formula

In addition to adding formal power series together, we can also multiply them. If $A(X) = \sum_{k=0}^{\infty} a_k x^k$ and $B(X) = \sum_{k=0}^{\infty} b_k x^k$, where a_k (resp. b_k) counts the number of objects of type A (resp. B) and size k, then

$$A(X)B(X) = C(X) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, and has a combinatorial interpretation as the number of objects of size n formed by taking an object of type A and concatenating it with an object of type B.

1.4 Connection between Linear Recurrences and Rational Generating Functions

The behavior we saw of the Fibonacci numbers and its generating function is an example of a more general theorem.

Theorem. (Theorem 4.1.1 of Enumerative Combinatorics 1 by Richard Stanley) Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be a fixed sequence of complex numbers, $d \geq 1$, and $\alpha_d \neq 0$. The following conditions on a function $f : \mathbb{N} \to \mathbb{C}$ are equivalent:

i) The generating function $F(X)$ equals

$$\sum_{n=0}^{\infty} f(n)x^n = \frac{P(x)}{Q(x)}$$

where $Q(x) = 1 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_d x^d$ and $P(x)$ is a polynomial of degree $< d$.

ii) For all $n \geq 0$, $f(n)$ satisfies the linear recurrence relation

$$f(n+d) + \alpha_1 f(n+d-1) + \alpha_2 f(n+d-2) + \ldots + \alpha_d f(n) = 0.$$

iii) For all $n \geq 0$,

$$f(n) = \sum_{i=1}^{k} P_i(n)\gamma_i^n$$

where

$$1 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_d x^d = \prod_{i=1}^{k} (1 - \gamma_i x)^{e_i}$$

with the γ_i's distinct and each $P_i(n)$ is a univariate polynomial (in n) of degree less than e_i.

Defintion. A generating function of the form $\frac{P(x)}{Q(x)}$ is a called a **rational generating function**.

18.312 Algebraic Combinatorics
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