Course 18.312: Algebraic Combinatorics

Lecture Notes # 7 Addendum by Gregg Musiker

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1 Möbius Function on Posets

This material closely follows selections from Chapter 3 of Enumerative Combinatorics 1 by Richard Stanley.

1.1 New Posets from Old

If P and Q are posets on disjoint sets, then the **disjoint union** (or **direct sum**) of P and Q is the poset $P \sqcup Q$ (also denoted as $P + Q$) on the union of sets P and Q such that $x \leq y$ in $P \sqcup Q$ if either (1) $x, y \in P$ and $x \leq y$ in P or (2) $x, y \in Q$ and $x \leq y$ in Q.

The **ordinal sum** of two posets $P \oplus Q$ (Q on top of P) is the poset whose elements are the set $P \sqcup Q$ with the property that $x \leq y$ in $P \oplus Q$ if either (1) $x \leq y$ in $P \sqcup Q$ or (2) $x \in P$ and $y \in Q$.

For example, an antichain on n elements is the direct sum of n copies of P_1, the poset on one element, while a chain on n elements is the ordinal sum of n copies of P_1. A poset that can be built up by the two operations of direct sum and ordinal sum from P_1 is known as a **series-parallel poset**.

Exercise 1: Find the unique four-element poset which is not a series-parallel poset.

The **direct product** of posets P and Q is the poset $P \times Q$ on the set $\{(x, y) : x \in P, y \in Q\}$ such that $(x, y) \leq (x', y')$ in $P \times Q$ if $x \leq x'$ in P and $y \leq y'$ in Q. One can draw the Hasse diagram for $P \times Q$ by first drawing the Hasse diagram of P and then replacing each element x of P with a copy Q_x ($\phi_x : Q_x \cong Q$) of the Hasse diagram of Q, and we connect element y of Q_x and y' of $Q_{x'}$ iff $\phi_{x'} \circ \phi_x^{-1}(y) = y'$

(i.e. y and y' correspond to the same element of Q) and x' covers x. Observe that $P \times Q \cong Q \times P$, and if P and Q are graded with rank-generating functions $F_P(q)$ and $F_Q(q)$, then

$$F_{P\times Q}(q) = F_P(q)F_Q(q).$$

See Section 4 of "Topics in Algebraic Combinatorics" by Richard Stanley for the definition of a graded poset.

The **dual** of a poset P is a poset P^* on the same set as P, but with order relations reversed. That is, $x \leq y$ in P iff $y \leq x$ in P^*.

An (induced) **subposet** Q of P is a subset of P such that for all $x, y \in Q$, $x \leq y$ in Q if and only if $x \leq y$ in P.

A (closed) **interval** $[x, y]$ in a poset P is a special kind of subposet defined as the set

$$[x, y] = \{z \in P : x \leq z \leq y\}.$$

For example, the interval $[x, x] = \{x\}$ and the empty set is not an interval. If every interval of P is finite, than P is called **locally finite**.

A (lower) **order ideal** $\mathcal{I}$ of a poset is a subposet which is closed under $\leq$, i.e., if $x \in \mathcal{I}$ and $y \leq x$, then $y \in \mathcal{I}$.

1.2 The Möbius Function in Number Theory

Before defining the Möbius function for more general posets, we discuss a family of posets arising from number theory.

For any positive integer n, we let D_n be the poset of all divisors of n. We say that $d_1 \leq d_2$ if $d_1 | d_2$ (d_1 divides d_2). For example, if $n = 12$ then D_{12} is given by the figure below.

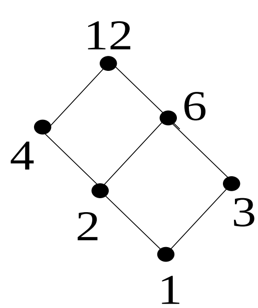

Figure 1: D_{12}'s Hasse Diagram

Notice that the poset D_n is graded with rank equal to the number of prime divisors (counting multiplicity) of n. The rank of a given element d in D_n is also equal to the number of prime divisors (counting multiplicity) of d.

To this poset, we define the (number theoretic) Möbius function to be

$$\hat{\mu}(n) = \begin{cases} 1 & \text{if } n \text{ is a squarefree positive integer with an } \textbf{even} \text{ number of distinct prime factors} \\ -1 & \text{if } n \text{ is a squarefree positive integer with an } \textbf{odd} \text{ number of distinct prime factors} \\ 0 & \text{if } n \text{ is not squarefree} \end{cases}.$$

The Möbius function arises in the formula for $\phi(n)$, the number of integers in $\{1, 2, \ldots, n\}$ which are relatively prime (share no common factor) with n:

$$\phi(n) = \sum_{d|n} \hat{\mu}(d) \left(\frac{n}{d}\right). \tag{1}$$

For example,

$$\begin{aligned} \phi(12) &= \#\{1, 5, 7, 11\} \\ &= \hat{\mu}(1)(12) + \hat{\mu}(2)(6) + \hat{\mu}(3)(4) + \hat{\mu}(4)(3) + \hat{\mu}(6)(2) + \hat{\mu}(12)(1) \\ &= 12 - 6 - 4 + 0 + 2 + 0 \\ &= 4. \end{aligned}$$

We will discuss how to define Möbius functions for other posets, and techniques for calculating this function. Applications will include the Möbius inversion formula which can be used to demonstrate that formula (1) is equivalent to

$$n = \sum_{d|n} \phi(d), \tag{2}$$

as well as the principle of inclusion-exclusion.

1.3 Möbius function of a poset

We define a map

$$\mu : P \times P \to \mathbb{Z}$$

by induction.

$$\begin{aligned} \mu(x, x) &= 1, \quad \text{for all } x \in P \\ \mu(x, y) &= -\sum_{x \leq z < y} \mu(x, z), \quad \text{for all } x < y \text{ in } P. \end{aligned}$$

An alternative way of expressing this definition is that $\mu(x, y)$ is the unique function such that $\mu(u, u) = 1$ and sums to zero on larger intervals (i.e. $\sum_{v\in[u,w]} \mu(u, v) = 0$ for all $u < v$ in P). This can be abbreviated as

$$\sum_{v\in[u,w]} \mu(u, v) = \delta_{u,w}.$$

Example 1/Exercise 2: Show that for the poset $P = D_n$, $\mu(1, d) = \hat{\mu}(d)$ for all d dividing n.

Example 2: We calculate $\mu(\emptyset, S)$ for subset S, an element of poset B_n. We begin with B_3, starting with $\mu(\emptyset, \emptyset) = 1$. We use this to calculate $\mu(\emptyset, \{i_1\}) = -1$ and then use diamond shape intervals (isomorphic to B_2) to show $\mu(\emptyset, \{i_1, i_2\}) = 1$. (Here $i_1, i_2 \in \{1, 2, 3\}$.) Finally, B_3 itself is an interval, and since the sum of all values of the Möbius function must sum to zero, the value $\mu(\emptyset, \{1, 2, 3\}) = -1$.

Exercise 3: Generalize this argument to B_n and show that for all $n \geq k \geq 1$,

$$\mu(\emptyset, \{i_1, i_2, \ldots, i_k\}) = (-1)^k \text{ in poset } B_n.$$

Proposition (Möbius inversion formula): Let P be a finite poset. (In fact this Proposition holds in more generality but we will not need this.) Let $f, g : P \to \mathbb{C}$. Then

$$g(x) = \sum_{y\geq x} f(y), \quad \text{for all } x \in P,$$

if and only if

$$f(x) = \sum_{y\geq x} g(y)\mu(x, y), \quad \text{for all } x \in P.$$

Application: The Principle of Inclusion-Exclusion

Say that we have four sets A, B, C, D, not necessarily disjoint, and we wish to count the number of elements in the union $A \cup B \cup C \cup D$. If we compute $|A| + |B| + |C| + |D|$, any element in the intersection of two of these sets is double-counted. However, if we compute

$$|A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|,$$

then any element in the intersection of three of these sets is now undercounted. (If $x \in A \cap B \cap C$ then x is counted three times in $|A| + |B| + |C|$ and removed three times by $-|A \cap B| - |A \cap C| - |B \cap C|$.) Thus we must add in the sum of the triple intersections, and lastly we subtract the size of the full intersection $|A \cap B \cap C \cap D|$.

In general, we have the formula

$$\left| \bigcup_{i \in [n]} A_i \right| = \sum_{S \subset [n]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|.$$

Exercise 4 Show that this formula is an application of Möbius inversion applied to the boolean poset B_n.