# LECTURE 2
# Properties of the intersection poset and graphical arrangements

## 2.1. Properties of the intersection poset

Let $\mathcal{A}$ be an arrangement in the vector space $V$. A *subarrangement* of $\mathcal{A}$ is a subset $\mathcal{B} \subseteq \mathcal{A}$. Thus $\mathcal{B}$ is also an arrangement in $V$. If $x \in L(\mathcal{A})$, define the subarrangement $\mathcal{A}_x \subseteq \mathcal{A}$ by

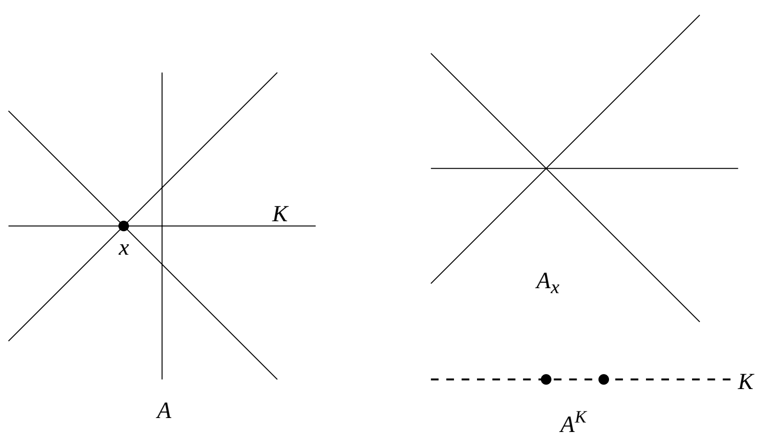
$$\mathcal{A}_x = \{H \in \mathcal{A} \,:\, x \subseteq H\}. \tag{6}$$

Also define an arrangement $\mathcal{A}^x$ in the affine subspace $x \in L(\mathcal{A})$ by

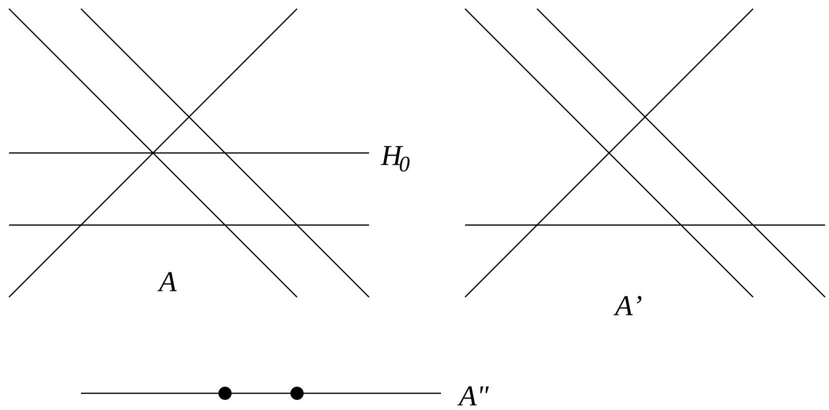
$$\mathcal{A}^x = \{x \cap H \neq \emptyset \,:\, H \in \mathcal{A} - \mathcal{A}_x\}.$$

Note that if $x \in L(\mathcal{A})$, then

$$\begin{aligned} L(\mathcal{A}_x) \cong \Lambda_x &:= \{y \in L(\mathcal{A}) \,:\, y \leq x\} \\ L(\mathcal{A}^x) \cong V_x &:= \{y \in L(\mathcal{A}) \,:\, y \geq x\} \end{aligned} \tag{7}$$




Choose $H_0 \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} - \{H_0\}$ and $\mathcal{A}'' = \mathcal{A}^{H_0}$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a *triple* of arrangements with *distinguished hyperplane* $H_0$.

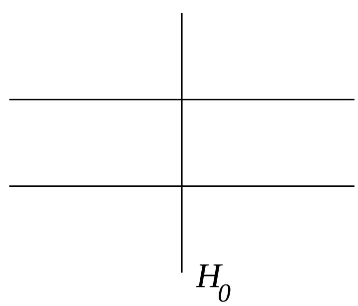



The main goal of this section is to give a formula in terms of $\chi_{\mathcal{A}}(t)$ for $r(\mathcal{A})$ and $b(\mathcal{A})$ when $K = \mathbb{R}$ (Theorem 2.5). We first establish recurrences for these two quantities.

**Lemma 2.1.** *Let* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *be a triple of real arrangements with distinguished hyperplane* $H_0$. *Then*

$$\begin{aligned} r(\mathcal{A}) &= r(\mathcal{A}') + r(\mathcal{A}'') \\ b(\mathcal{A}) &= \begin{cases} b(\mathcal{A}') + b(\mathcal{A}''), & \text{if } \operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}') \\ 0, & \text{if } \operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}') + 1. \end{cases} \end{aligned}$$

Note. If $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}')$, then also $\operatorname{rank}(\mathcal{A}) = 1 + \operatorname{rank}(\mathcal{A}'')$. The figure below illustrates the situation when $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}') + 1$.




**Proof.** Note that $r(\mathcal{A})$ equals $r(\mathcal{A}')$ plus the number of regions of $\mathcal{A}'$ cut into two regions by $H_0$. Let $R'$ be such a region of $\mathcal{A}'$. Then $R' \cap H_0 \in \mathcal{R}(\mathcal{A}'')$. Conversely, if $R'' \in \mathcal{R}(\mathcal{A}'')$ then points near $R''$ on either side of $H_0$ belong to the same region $R' \in \mathcal{R}(\mathcal{A}')$, since any $H \in \mathcal{R}(\mathcal{A}')$ separating them would intersect $R''$. Thus $R'$ is cut in two by $H_0$. We have established a bijection between regions of $\mathcal{A}'$ cut into two by $H_0$ and regions of $\mathcal{A}''$, establishing the first recurrence.

The second recurrence is proved analogously; the details are omitted. $\square$

We now come to the fundamental recursive property of the characteristic polynomial.

**Lemma 2.2.** *(Deletion-Restriction) Let* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ *be a triple of real arrangements. Then*

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t).$$

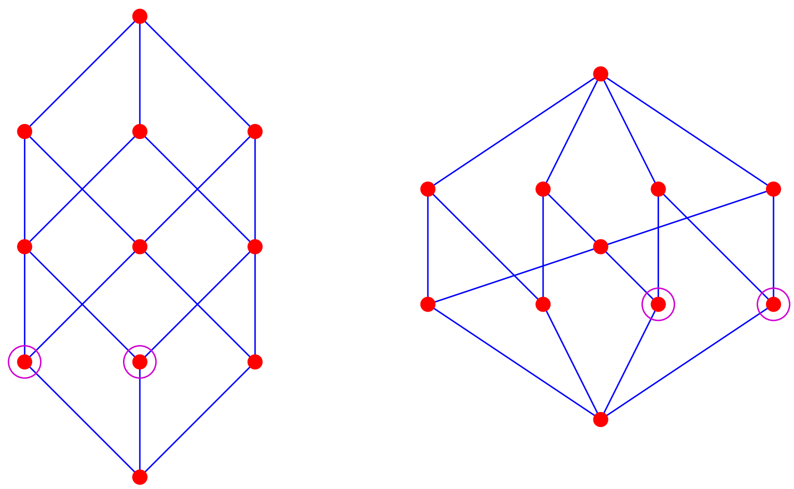


**Figure 1.** Two non-lattices

For the proof of this lemma, we will need some tools. (A more elementary proof could be given, but the tools will be useful later.)

Let $P$ be a poset. An *upper bound* of $x, y \in P$ is an element $z \in P$ satisfying $z \geq x$ and $z \geq y$. A *least upper bound* or *join* of $x$ and $y$, denoted $x \vee y$, is an upper bound $z$ such that $z \leq z'$ for all upper bounds $z'$. Clearly if $x \vee y$ exists, then it is unique. Similarly define a *lower bound* of $x$ and $y$, and a *greatest lower bound* or *meet*, denoted $x \wedge y$. A *lattice* is a poset $L$ for which any two elements have a meet and join. A *meet-semilattice* is a poset $P$ for which any two elements have a meet. Dually, a *join-semilattice* is a poset $P$ for which any two elements have a join. Figure 1 shows two non-lattices, with a pair of elements circled which don't have a join.

**Lemma 2.3.** *A finite meet-semilattice $L$ with a unique maximal element $\hat{1}$ is a lattice. Dually, a finite join-semilattice $L$ with a unique minimal element $\hat{0}$ is a lattice.*

**Proof.** Let $L$ be a finite meet-semilattice. If $x, y \in L$ then the set of upper bounds of $x, y$ is nonempty since $\hat{1}$ is an upper bound. Hence

$$x \vee y = \bigwedge_{\substack{z \geq x \\ z \geq y}} z.$$

The statement for join-semilattices is by "duality," i.e., interchanging $\leq$ with $\geq$, and $\wedge$ with $\vee$. $\square$

The reader should check that Lemma 2.3 need not hold for *infinite* semilattices.

**Proposition 2.3.** *Let $\mathcal{A}$ be an arrangement. Then $L(\mathcal{A})$ is a meet-semilattice. In particular, every interval $[x, y]$ of $L(\mathcal{A})$ is a lattice. Moreover, $L(\mathcal{A})$ is a lattice if and only if $\mathcal{A}$ is central.*

**Proof.** If $\bigcap_{H \in \mathcal{A}} H = \emptyset$, then adjoin $\emptyset$ to $L(\mathcal{A})$ as the unique maximal element, obtaining the augmented intersection poset $L'(\mathcal{A})$. In $L'(\mathcal{A})$ it is clear that $x \vee y = x \cap y$. Hence $L'(\mathcal{A})$ is a join-semilattice. Since it has a $\hat{0}$, it is a lattice by Lemma 2.3.

Since $L(\mathcal{A}) = L'(\mathcal{A})$ or $L(\mathcal{A}) = L'(\mathcal{A}) - \{\hat{1}\}$, it follows that $L(\mathcal{A})$ is always a meet-semilattice, and is a lattice if $\mathcal{A}$ is central. If $\mathcal{A}$ isn't central, then $\bigvee_{x\in L(\mathcal{A})} x$ does not exist, so $L(\mathcal{A})$ is not a lattice. □

We now come to a basic formula for the Möbius function of a lattice.

**Theorem 2.2.** *(the Cross-Cut Theorem) Let $L$ be a finite lattice. Let $X$ be a subset of $L$ such that $\hat{0} \notin X$, and such that if $y \in L$, $y \neq \hat{0}$, then some $x \in X$ satisfies $x \leq y$. Let $N_k$ be the number of $k$-element subsets of $X$ with join $\hat{1}$. Then*

$$\mu_L(\hat{0}, \hat{1}) = N_0 - N_1 + N_2 - \cdots .$$

We will prove Theorem 2.2 by an algebraic method. Such a sophisticated proof is unnecessary, but the machinery we develop will be used later (Theorem 4.13). Let $L$ be a finite lattice and $K$ a field. The *Möbius algebra* of $L$, denoted $A(L)$, is the semigroup algebra of $L$ over $K$ with respect to the operation $\vee$. (Sometimes the operation is taken to be $\wedge$ instead of $\vee$, but for our purposes, $\vee$ is more convenient.) In other words, $A(L) = KL$ (the vector space with basis $L$) as a vector space. If $x, y \in L$ then we define $xy = x \vee y$. Multiplication is extended to all of $A(L)$ by bilinearity (or distributivity). Algebraists will recognize that $A(L)$ is a finite-dimensional commutative algebra with a basis of idempotents, and hence is isomorphic to $K^{\#L}$ (as an algebra). We will show this by exhibiting an explicit isomorphism $A(L) \xrightarrow{\cong} K^{\#L}$. For $x \in L$, define

$$\sigma_x = \sum_{y\geq x} \mu(x, y) y \in A(L), \tag{8}$$

where $\mu$ denotes the Möbius function of $L$. Thus by the Möbius inversion formula,

$$x = \sum_{y\geq x} \sigma_y, \text{ for all } x \in L. \tag{9}$$

Equation (9) shows that the $\sigma_x$'s span $A(L)$. Since $\#\{\sigma_x : x \in L\} = \#L = \dim A(L)$, it follows that the $\sigma_x$'s form a basis for $A(L)$.

**Theorem 2.3.** *Let $x, y \in L$. Then $\sigma_x\sigma_y = \delta_{xy}\sigma_x$, where $\delta_{xy}$ is the Kronecker delta. In other words, the $\sigma_x$'s are orthogonal idempotents. Hence*

$$A(L) = \bigoplus_{x\in L} K \cdot \sigma_x \quad \textit{(algebra direct sum).}$$

**Proof.** Define a $K$-algebra $A'(L)$ with basis $\{\sigma'_x : x \in L\}$ and multiplication $\sigma'_x\sigma'_y = \delta_{xy}\sigma'_x$. For $x \in L$ set $x' = \sum_{s\geq x} \sigma'_s$. Then

$$\begin{aligned} x'y' &= \left(\sum_{s\geq x} \sigma'_s\right)\left(\sum_{t\geq y} \sigma'_t\right) \\ &= \sum_{\substack{s\geq x \\ s\geq y}} \sigma'_s \\ &= \sum_{s\geq x\vee y} \sigma'_s \\ &= (x \vee y)'. \end{aligned}$$

Hence the linear transformation $\varphi : A(L) \to A'(L)$ defined by $\varphi(x) = x'$ is an algebra isomorphism. Since $\varphi(\sigma_x) = \sigma'_x$, it follows that $\sigma_x\sigma_y = \delta_{xy}\sigma_x$. □

NOTE. The algebra $A(L)$ has a multiplicative identity, viz., $1 = \hat{0} = \sum_{x \in L} \sigma_x$.

**Proof of Theorem 2.2.** Let $\mathrm{char}(K) = 0$, e.g., $K = \mathbb{Q}$. For any $x \in L$, we have in $A(L)$ that

$$\hat{0} - x = \sum_{y \geq \hat{0}} \sigma_y - \sum_{y \geq x} \sigma_y = \sum_{y \not\geq x} \sigma_y.$$

Hence by the orthogonality of the $\sigma_y$'s we have

$$\prod_{x \in X} (\hat{0} - x) = \sum_{y} \sigma_y,$$

where $y$ ranges over all elements of $L$ satisfying $y \not\geq x$ for all $x \in X$. By hypothesis, the only such element is $\hat{0}$. Hence

$$\prod_{x \in X} (\hat{0} - x) = \sigma_{\hat{0}}.$$

If we now expand both sides as linear combinations of elements of $L$ and equate coefficients of $\hat{1}$, the result follows. $\square$

NOTE. In a finite lattice $L$, an *atom* is an element covering $\hat{0}$. Let $T$ be the set of atoms of $L$. Then a set $X \subseteq L - \{\hat{0}\}$ satisfies the hypotheses of Theorem 2.2 if and only if $T \subseteq X$. Thus the simplest choice of $X$ is just $X = T$.

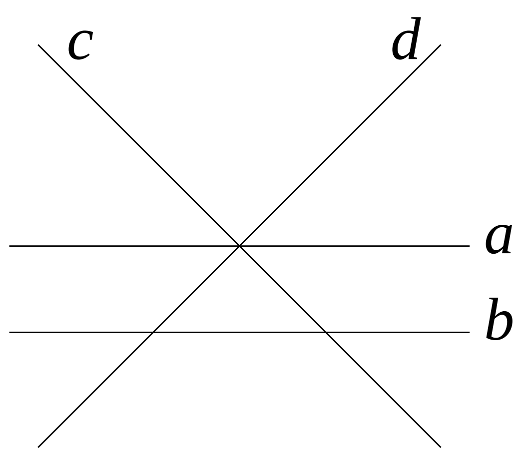
**Example 2.5.** Let $L = B_n$, the boolean algebra of all subsets of $[n]$. Let $X = T = \{\{i\} : i \in [n]\}$. Then $N_0 = N_1 = \cdots = N_{n-1} = 0$, $N_n = 1$. Hence $\mu(\hat{0}, \hat{1}) = (-1)^n$, agreeing with Proposition 1.2.

We will use the Crosscut Theorem to obtain a formula for the characteristic polynomial of an arrangement $\mathcal{A}$. Extending slightly the definition of a central arrangement, call any subset $\mathcal{B}$ of $\mathcal{A}$ *central* if $\bigcap_{H \in \mathcal{B}} H \neq \emptyset$. The following result is due to Hassler Whitney for linear arrangements. Its easy extension to arbitrary arrangements appears in [**13**, Lemma 2.3.8].

**Theorem 2.4.** *(Whitney's theorem) Let $\mathcal{A}$ be an arrangement in an $n$-dimensional vector space. Then*

$$\chi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} t^{n - \mathrm{rank}(\mathcal{B})}. \tag{10}$$

**Example 2.6.** Let $\mathcal{A}$ be the arrangement in $\mathbb{R}^2$ shown below.



The following table shows all central subsets $\mathcal{B}$ of $\mathcal{A}$ and the values of $\#\mathcal{B}$ and $\mathrm{rank}(\mathcal{B})$.

| $\mathcal{B}$ | $\#\mathcal{B}$ | $\mathrm{rank}(\mathcal{B})$ |
|---|---|---|
| $\emptyset$ | 0 | 0 |
| $a$ | 1 | 1 |
| $b$ | 1 | 1 |
| $c$ | 1 | 1 |
| $d$ | 1 | 1 |
| $ac$ | 2 | 2 |
| $ad$ | 2 | 2 |
| $bc$ | 2 | 2 |
| $bd$ | 2 | 2 |
| $cd$ | 2 | 2 |
| $acd$ | 3 | 2 |

It follows that $\chi_{\mathcal{A}}(t) = t^2 - 4t + (5-1) = t^2 - 4t + 4$.

**Proof of Theorem 2.4.** Let $z \in L(\mathcal{A})$. Let

$$\Lambda_z = \{x \in L(\mathcal{A}) : x \leq z\},$$

the *principal order ideal* generated by $z$. Recall the definition

$$\mathcal{A}_z = \{H \in \mathcal{A} : H \leq z \text{ (i.e., } z \subseteq H)\}.$$

By the Crosscut Theorem (Theorem 2.2), we have

$$\mu(z) = \sum_k (-1)^k N_k(z),$$

where $N_k(z)$ is the number of $k$-subsets of $\mathcal{A}_z$ with join $z$. In other words,

$$\mu(z) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A}_z \\ z = \bigcap_{H \in \mathcal{B}} H}} (-1)^{\#\mathcal{B}}.$$

Note that $z = \bigcap_{H \in \mathcal{B}} H$ implies that $\mathrm{rank}(\mathcal{B}) = n - \dim z$. Now multiply both sides by $t^{\dim(z)}$ and sum over $z$ to obtain equation (10). $\square$

We have now assembled all the machinery necessary to prove the Deletion-Restriction Lemma (Lemma 2.2) for $\chi_{\mathcal{A}}(t)$.

**Proof of Lemma 2.2.** Let $H_0 \in \mathcal{A}$ be the hyperplane defining the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$. Split the sum on the right-hand side of (10) into two sums, depending on whether $H_0 \notin \mathcal{B}$ or $H_0 \in \mathcal{B}$. In the former case we get

$$\sum_{\substack{H_0 \notin \mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} t^{n - \mathrm{rank}(\mathcal{B})} = \chi_{\mathcal{A}'}(t).$$

In the latter case, set $\mathcal{B}_1 = (\mathcal{B} - \{H_0\})^{H_0}$, a central arrangement in $H_0 \cong K^{n-1}$ and a subarrangement of $\mathcal{A}^{H_0} = \mathcal{A}''$. Since $\#\mathcal{B}_1 = \#\mathcal{B} - 1$ and $\mathrm{rank}(\mathcal{B}_1) = \mathrm{rank}(\mathcal{B}) - 1$, we get

$$\begin{aligned} \sum_{\substack{H_0 \in \mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} t^{n - \mathrm{rank}(\mathcal{B})} &= \sum_{\mathcal{B}_1 \in \mathcal{A}''} (-1)^{\#\mathcal{B}_1 + 1} t^{(n-1) - \mathrm{rank}(\mathcal{B}_1)} \\ &= -\chi_{\mathcal{A}''}(t), \end{aligned}$$

and the proof follows. $\square$

## 2.2. The number of regions

The next result is perhaps the first major theorem in the subject of hyperplane arrangements, due to Thomas Zaslavsky in 1975.

**Theorem 2.5.** *Let $\mathcal{A}$ be an arrangement in an $n$-dimensional real vector space. Then*

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1) \tag{11}$$

$$b(\mathcal{A}) = (-1)^{\operatorname{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1). \tag{12}$$

**First proof.** Equation (11) holds for $\mathcal{A} = \emptyset$, since $r(\emptyset) = 1$ and $\chi_\emptyset(t) = t^n$. By Lemmas 2.1 and 2.2, both $r(\mathcal{A})$ and $(-1)^n \chi_{\mathcal{A}}(-1)$ satisfy the same recurrence, so the proof follows.

Now consider equation (12). Again it holds for $\mathcal{A} = \emptyset$ since $b(\emptyset) = 1$. (Recall that $b(\mathcal{A})$ is the number of *relatively* bounded regions. When $\mathcal{A} = \emptyset$, the entire ambient space $\mathbb{R}^n$ is relatively bounded.) Now

$$\chi_{\mathcal{A}}(1) = \chi_{\mathcal{A}'}(1) - \chi_{\mathcal{A}''}(1).$$

Let $d(\mathcal{A}) = (-1)^{\operatorname{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1)$. If $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}') = \operatorname{rank}(\mathcal{A}'') + 1$, then $d(\mathcal{A}) = d(\mathcal{A}') + d(\mathcal{A}'')$. If $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{A}') + 1$ then $b(\mathcal{A}) = 0$ [why?] and $L(\mathcal{A}') \cong L(\mathcal{A}'')$ [why?]. Thus from Lemma 2.2 we have $d(\mathcal{A}) = 0$. Hence in all cases $b(\mathcal{A})$ and $d(\mathcal{A})$ satisfy the same recurrence, so $b(\mathcal{A}) = d(\mathcal{A})$. □

**Second proof.** Our second proof of Theorem 2.5 is based on Möbius inversion and some instructive topological considerations. For this proof we assume basic knowledge of the Euler characteristic $\psi(\Delta)$ of a topological space $\Delta$. (Standard notation is $\chi(\Delta)$, but this would cause too much confusion with the characteristic polynomial.) In particular, if $\Delta$ is suitably decomposed into cells with $f_i$ $i$-dimensional cells, then

$$\psi(\Delta) = f_0 - f_1 + f_2 - \cdots . \tag{13}$$

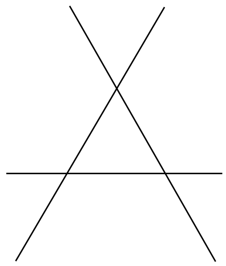
We take (13) as the definition of $\psi(\Delta)$. For "nice" spaces and decompositions, it is independent of the decomposition. In particular, $\psi(\mathbb{R}^n) = (-1)^n$. Write $\bar{R}$ for the closure of a region $R \in \mathcal{R}(\mathcal{A})$.

**Definition 2.4.** A (closed) *face* of a real arrangement $\mathcal{A}$ is a set $\emptyset \neq F = \bar{R} \cap x$, where $R \in \mathcal{R}(\mathcal{A})$ and $x \in L(\mathcal{A})$.

If we regard $\bar{R}$ as a convex polyhedron (possibly unbounded), then a face of $\mathcal{A}$ is just a face of some $\bar{R}$ in the usual sense of the face of a polyhedron, i.e., the intersection of $\bar{R}$ with a supporting hyperplane. In particular, each $\bar{R}$ is a face of $\mathcal{A}$. The *dimension* of a face $F$ is defined by

$$\dim(F) = \dim(\operatorname{aff}(F)),$$

where $\operatorname{aff}(F)$ denotes the affine span of $F$. A *$k$-face* is a $k$-dimensional face of $\mathcal{A}$. For instance, the arrangement below has three 0-faces (vertices), nine 1-faces, and seven 2-faces (equivalently, seven regions). Hence $\psi(\mathbb{R}^2) = 3 - 9 + 7 = 1$.



Write $\mathcal{F}(\mathcal{A})$ for the set of faces of $\mathcal{A}$, and let relint denote relative interior. Then

$$\mathbb{R}^n = \bigsqcup_{F \in \mathcal{F}(\mathcal{A})} \operatorname{relint}(F),$$

where $\bigsqcup$ denotes disjoint union. If $f_k(\mathcal{A})$ denotes the number of $k$-faces of $\mathcal{A}$, it follows that

$$(-1)^n = \psi(\mathbb{R}^n) = f_0(\mathcal{A}) - f_1(\mathcal{A}) + f_2(\mathcal{A}) - \cdots .$$

Every $k$-face is a region of exactly one $\mathcal{A}^y$ for $y \in L(\mathcal{A})$. Hence

$$f_k(\mathcal{A}) = \sum_{\substack{y \in L(\mathcal{A}) \\ \dim(y)=k}} r(\mathcal{A}^y).$$

Multiply by $(-1)^k$ and sum over $k$ to get

$$(-1)^n = \psi(\mathbb{R}^n) = \sum_{y \in L(\mathcal{A})} (-1)^{\dim(y)} r(\mathcal{A}^y).$$

Replacing $\mathbb{R}^n$ by $x \in L(\mathcal{A})$ gives

$$(-1)^{\dim(x)} = \psi(x) = \sum_{\substack{y \in L(\mathcal{A}) \\ y \geq x}} (-1)^{\dim(y)} r(\mathcal{A}^y).$$

Möbius inversion yields

$$(-1)^{\dim(x)} r(\mathcal{A}^x) = \sum_{\substack{y \in L(\mathcal{A}) \\ y \geq x}} (-1)^{\dim(y)} \mu(x,y).$$

Putting $x = \mathbb{R}^n$ gives

$$(-1)^n r(\mathcal{A}) = \sum_{y \in L(\mathcal{A})} (-1)^{\dim(y)} \mu(y) = \chi_{\mathcal{A}}(-1),$$

thereby proving (11).

The relatively bounded case (equation (12)) is similar, but with one technical complication. We may assume that $\mathcal{A}$ is essential, since $b(\mathcal{A}) = b(\operatorname{ess}(\mathcal{A}))$ and

$$\chi_{\mathcal{A}}(t) = t^{\dim(\mathcal{A}) - \dim(\operatorname{ess}(\mathcal{A}))} \chi_{\operatorname{ess}(\mathcal{A})}(t).$$

In this case, the relatively bounded regions are actually bounded. Let

$$\begin{aligned} \mathcal{F}_b(\mathcal{A}) &= \{F \in \mathcal{F}(\mathcal{A}) \,:\, F \text{ is relatively bounded}\} \\ \Gamma &= \bigcup_{F \in \mathcal{F}_b(\mathcal{A})} F. \end{aligned}$$

The difficulty lies in computing $\psi(\Gamma)$. Zaslavsky conjectured in 1975 that $\Gamma$ is star-shaped, i.e., there exists $x \in \Gamma$ such that for every $y \in \Gamma$, the line segment

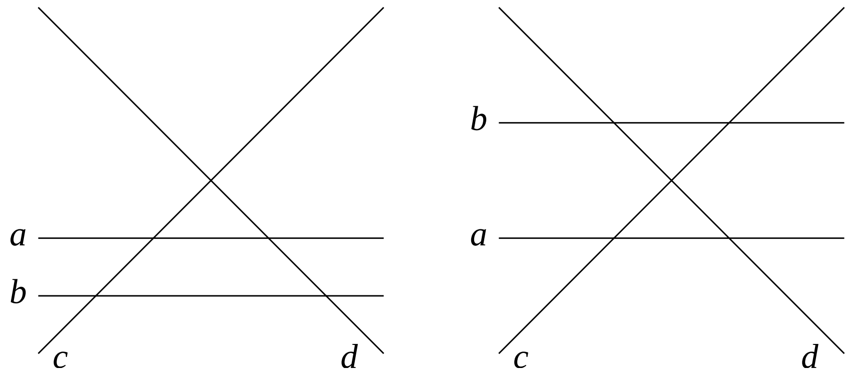



**Figure 2.** Two arrangements with the same intersection poset

joining $x$ and $y$ lies in $\Gamma$. This would imply that $\Gamma$ is contractible, and hence (since $\Gamma$ is compact when $\mathcal{A}$ is essential) $\psi(\Gamma) = 1$. A counterexample to Zaslavsky's conjecture appears as an exercise in [**5**, Exer. 4.29], but nevertheless Björner and Ziegler showed that $\Gamma$ is indeed contractible. (See [**5**, Thm. 4.5.7(b)] and Lecture 1, Exercise 7.) The argument just given for $r(\mathcal{A})$ now carries over *mutatis mutandis* to $b(\mathcal{A})$. There is also a direct argument that $\psi(\Gamma) = 1$, circumventing the need to show that $\Gamma$ is contractible. We will omit proving here that $\psi(\Gamma) = 1$. $\square$

**Corollary 2.1.** *Let $\mathcal{A}$ be a real arrangement. Then $r(\mathcal{A})$ and $b(\mathcal{A})$ depend only on $L(\mathcal{A})$.*

Figure 2 shows two arrangements in $\mathbb{R}^2$ with different "face structure" but the same $L(\mathcal{A})$. The first arrangement has for instance one triangular and one quadrilateral face, while the second has two triangular faces. Both arrangements, however, have ten regions and two bounded regions.

We now give two basic examples of arrangements and the computation of their characteristic polynomials.

**Proposition 2.4.** *(general position) Let $\mathcal{A}$ be an $n$-dimensional arrangement of $m$ hyperplanes in general position. Then*

$$\chi_{\mathcal{A}}(t) = t^n - mt^{n-1} + \binom{m}{2}t^{n-2} - \cdots + (-1)^n\binom{m}{n}.$$

*In particular, if $\mathcal{A}$ is a real arrangement, then*

$$\begin{aligned} r(\mathcal{A}) &= 1 + m + \binom{m}{2} + \cdots + \binom{m}{n} \\ b(\mathcal{A}) &= (-1)^n\left(1 - m + \binom{m}{2} - \cdots + (-1)^n\binom{m}{n}\right) \\ &= \binom{m-1}{n}. \end{aligned}$$

**Proof.** Every $\mathcal{B} \subseteq \mathcal{A}$ with $\#\mathcal{B} \leq n$ defines an element $x_{\mathcal{B}} = \bigcap_{H\in\mathcal{B}} H$ of $L(\mathcal{A})$. Hence $L(\mathcal{A})$ is a *truncated boolean algebra*:

$$L(\mathcal{A}) \cong \{S \subseteq [m] : \#S \leq n\},$$

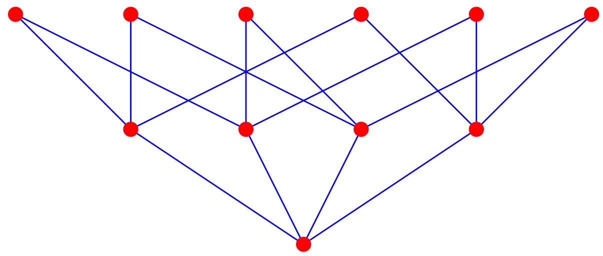


**Figure 3.** The truncated boolean algebra of rank 2 with four atoms

ordered by inclusion. Figure 3 shows the case $n = 2$ and $m = 4$, i.e., four lines in general position in $\mathbb{R}^2$. If $x \in L(\mathcal{A})$ and $\mathrm{rk}(x) = k$, then $[\hat{0}, x] \cong B_k$, a boolean algebra of rank $k$. By equation (4) there follows $\mu(x) = (1)^k$. Hence

$$\begin{aligned}\chi_{\mathcal{A}}(t) &= \sum_{\substack{S \subseteq [m] \\ \#S \leq n}} (-1)^{\#S} t^{n-\#S} \\ &= t^n - mt^{n-1} + \cdots + (-1)^n \binom{m}{n}. \quad \square\end{aligned}$$

NOTE. Arrangements whose hyperplanes are in general position were formerly called *free arrangements*. Now, however, free arrangements have another meaning discussed in the note following Example 4.11.

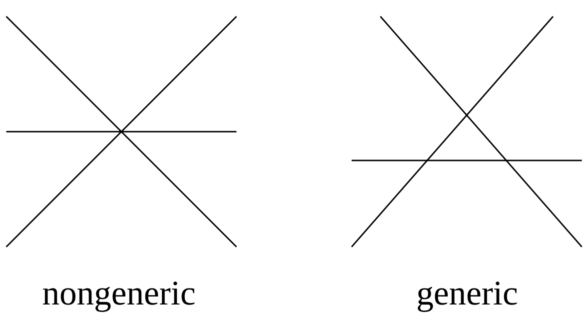
Our second example concerns generic translations of the hyperplanes of a linear arrangement. Let $L_1, \dots, L_m$ be linear forms, not necessarily distinct, in the variables $v = (v_1, \dots, v_n)$ over the field $K$. Let $\mathcal{A}$ be defined by

$$L_1(v) = a_1, \dots, L_m(v) = a_m,$$

where $a_1, \dots, a_m$ are *generic* elements of $K$. This means if $H_i = \ker(L_i(v) - a_i)$, then

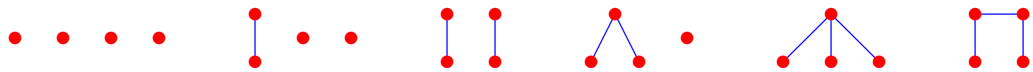
$$H_{i_1} \cap \cdots \cap H_{i_k} \neq \emptyset \;\Leftrightarrow\; L_{i_1}, \dots, L_{i_k} \text{ are linearly independent.}$$

For instance, if $K = \mathbb{R}$ and $L_1, \dots, L_m$ are defined over $\mathbb{Q}$, then $a_1, \dots, a_m$ are generic whenever they are linearly independent over $\mathbb{Q}$.




It follows that if $x = H_{i_1} \cap \cdots \cap H_{i_k} \in L(\mathcal{A})$, then $[\hat{0}, x] \cong B_k$. Hence

$$\chi_{\mathcal{A}}(t) = \sum_{\mathcal{B}} (-1)^{\#\mathcal{B}} t^{n-\#\mathcal{B}},$$



1 6 3 12 4 12

**Figure 4.** The forests on four vertices

where $\mathcal{B}$ ranges over all linearly independent subsets of $\mathcal{A}$. (We say that a set of hyperplanes are *linearly independent* if their normals are linearly independent.) Thus $\chi_{\mathcal{A}}(t)$, or more precisely $(-t)^n\chi_{\mathcal{A}}(-1/t)$, is the generating function for linearly independent subsets of $L_1, \dots, L_m$ according to their number of elements. For instance, if $\mathcal{A}$ is given by Figure 2 (either arrangement) then the linearly independent subsets of hyperplanes are $\emptyset, a, b, c, d, ac, ad, bc, bd, cd$, so $\chi_{\mathcal{A}}(t) = t^2 - 4t + 5$.

Consider the more interesting example $x_i - x_j = a_{ij}$, $1 \le i < j \le n$, where the $a_{ij}$ are generic. We could call this arrangement the *generic braid arrangement* $\mathcal{G}_n$. Identify the hyperplane $x_i - x_j = a_{ij}$ with the edge $ij$ on the vertex set $[n]$. Thus a subset $\mathcal{B} \subseteq \mathcal{G}_n$ corresponds to a simple graph $G_{\mathcal{B}}$ on $[n]$. ("Simple" means that there is at most one edge between any two vertices, and no edge from a vertex to itself.) It is easy to see that $\mathcal{B}$ is linearly independent if and only if the graph $G_{\mathcal{B}}$ has no cycles, i.e., is a *forest*. Hence we obtain the interesting formula

$$\chi_{\mathcal{G}_n}(t) = \sum_F (-1)^{e(F)} t^{n-e(F)}, \tag{14}$$

where $F$ ranges over all forests on $[n]$ and $e(F)$ denotes the number of edges of $F$. For instance, the isomorphism types of forests (with the number of distinct labelings written below the forest) on four vertices are given by Figure 4. Hence

$$\chi_{\mathcal{G}_4}(t) = t^4 - 6t^3 + 15t^2 - 16t.$$

Equation (11) can be rewritten as

$$r(\mathcal{A}) = \sum_{x \in L(\mathcal{A})} (-1)^{\operatorname{rk}(x)} \mu(x).$$

(Theorem 3.10 will show that $(-1)^{\operatorname{rk}(x)}\mu(x) > 0$, so we could also write $|\mu(x)|$ for this quantity.) It is easy to extend this result to count faces of $\mathcal{A}$ of all dimensions, not just the top dimension $n$. Let $f_k(\mathcal{A})$ denote the number of $k$-faces of the real arrangement $\mathcal{A}$.

**Theorem 2.6.** *We have*

$$f_k(\mathcal{A}) = \sum_{\substack{x \le y \text{ in } L(\mathcal{A}) \\ \dim(x) = k}} (-1)^{\dim(x) - \dim(y)} \mu(x, y) \tag{15}$$

$$= \sum_{\substack{x \le y \text{ in } L(\mathcal{A}) \\ \dim(x) = k}} |\mu(x, y)|. \tag{16}$$

**Proof.** As mentioned above, every face $F$ is a region of a unique $\mathcal{A}^x$ for $x \in L(\mathcal{A})$, viz., $x = \operatorname{aff}(F)$. In particular, $\dim(F) = \dim(x)$. Hence if $\dim(F) = k$, then $r(\mathcal{A}^x)$ is the number of $k$-faces of $\mathcal{A}$ contained in $x$. By Theorem 2.5 and equation (7) we get

$$r(\mathcal{A}^x) = \sum_{y \ge x} (-1)^{\dim(y) - \dim(x)} \mu(x, y),$$

where we are dealing with the poset $L(\mathcal{A})$. Summing over all $x \in L(\mathcal{A})$ of dimension $k$ yields (15), and (16) then follows from Theorem (3.10) below. $\square$

### 2.3. Graphical arrangements

There are close connections between certain invariants of a graph $G$ and an associated arrangement $\mathcal{A}_G$. Let $G$ be a simple graph on the vertex set $[n]$. Let $E(G)$ denote the set of edges of $G$, regarded as two-element subsets of $[n]$. Write $ij$ for the edge $\{i, j\}$.

**Definition 2.5.** The *graphical arrangement* $\mathcal{A}_G$ in $K^n$ is the arrangement

$$x_i - x_j = 0, \ ij \in E(G).$$

Thus a graphical arrangement is simply a subarrangement of the braid arrangement $\mathcal{B}_n$. If $G = K_n$, the *complete graph* on $[n]$ (with all possible edges $ij$), then $\mathcal{A}_{K_n} = \mathcal{B}_n$.

**Definition 2.6.** A *coloring* of a graph $G$ on $[n]$ is a map $\kappa : [n] \to \mathbb{P}$. The coloring $\kappa$ is *proper* if $\kappa(i) \neq \kappa(j)$ whenever $ij \in E(G)$. If $q \in \mathbb{P}$ then let $\chi_G(q)$ denote the number of proper colorings $\kappa : [n] \to [q]$ of $G$, i.e., the number of proper colorings of $G$ whose colors come from $1, 2, \ldots, q$. The function $\chi_G$ is called the *chromatic polynomial* of $G$.

For instance, suppose that $G$ is the complete graph $K_n$. A proper coloring $\kappa : [n] \to [q]$ is obtained by choosing a vertex, say 1, and coloring it in $q$ ways. Then choose another vertex, say 2, and color it in $q - 1$ ways, etc., obtaining

$$\chi_{K_n}(q) = q(q-1)\cdots(q-n+1). \tag{17}$$

A similar argument applies to the graph $G$ of Figure 5. There are $q$ ways to color vertex 1, then $q - 1$ to color vertex 2, then $q - 1$ to color vertex 3, etc., obtaining

$$\begin{aligned} \chi_G(q) &= q(q-1)(q-1)(q-2)(q-1)(q-1)(q-2)(q-2)(q-3) \\ &= q(q-1)^4(q-2)^3(q-3). \end{aligned}$$

Unlike the case of the complete graph, in order to obtain this nice product formula one factor at a time only certain orderings of the vertices are suitable. It is not always possible to evaluate the chromatic polynomials "one vertex at a time." For instance, let $H$ be the 4-cycle of Figure 5. If a proper coloring $\kappa : [4] \to [q]$ satisfies $\kappa(1) = \kappa(3)$, then there are $q$ choices for $\kappa(1)$, then $q - 1$ choices each for $\kappa(2)$ and $\kappa(4)$. On the other hand, if $\kappa(1) \neq \kappa(3)$, then there are $q$ choices for $\kappa(1)$, then $q - 1$ choices for $\kappa(3)$, and then $q - 2$ choices each for $\kappa(2)$ and $\kappa(4)$. Hence

$$\begin{aligned} \chi_H(q) &= q(q-1)^2 + q(q-1)(q-2)^2 \\ &= q(q-1)(q^2 - 3q + 3). \end{aligned}$$

For further information on graphs whose chromatic polynomial *can* be evaluated one vertex at a time, see Corollary 4.10 and the note following it.

It is easy to see directly that $\chi_G(q)$ is a polynomial function of $q$. Let $e_i(G)$ denote the number of *surjective* proper colorings $\kappa : [n] \to [i]$ of $G$. We can choose an arbitrary proper coloring $\kappa : [n] \to [q]$ by first choosing the size $i = \#\kappa([n])$ of its image in $\binom{q}{i}$ ways, and then choose $\kappa$ in $e_i$ ways. Hence

$$\chi_G(q) = \sum_{i=0}^{n} e_i \binom{q}{i}. \tag{18}$$

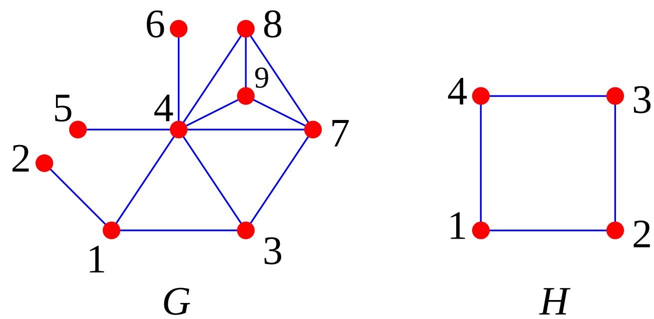
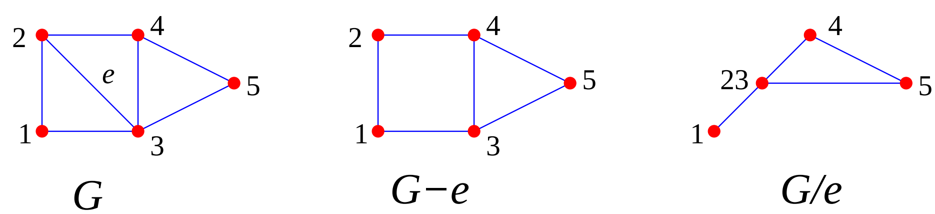



**Figure 5.** Two graphs

Since $\binom{q}{i} = q(q-1)\cdots(q-i+1)/i!$, a polynomial in $q$ (of degree $i$), we see that $\chi_G(q)$ is a polynomial. We therefore write $\chi_G(t)$, where $t$ is an indeterminate. Moreover, any surjection (= bijection) $\kappa : [n] \to [n]$ is proper. Hence $e_n = n!$. It follows from equation (18) that $\chi_G(t)$ is monic of degree $n$. Using more sophisticated methods we will later derive further properties of the coefficients of $\chi_G(t)$.

**Theorem 2.7.** *For any graph $G$, we have $\chi_{\mathcal{A}_G}(t) = \chi_G(t)$.*

**First proof.** The first proof is based on deletion-restriction (which in the context of graphs is called *deletion-contraction*). Let $e = ij \in E(G)$. Let $G - e$ (also denoted $G \backslash e$) denote the graph $G$ with edge $e$ deleted, and let $G/e$ denote $G$ with the edge $e$ contracted to a point and all multiple edges replaced by a single edge (i.e., whenever there is more than one edge between two vertices, replace these edges by a single edge). (In some contexts we want to keep track of multiple edges, but they are irrelevant in regard to proper colorings.)




Let $H_0 \in \mathcal{A} = \mathcal{A}_G$ be the hyperplane $x_i = x_j$. It is clear that $\mathcal{A} - \{H_0\} = \mathcal{A}_{G-e}$. We claim that

$$\mathcal{A}^{H_0} = \mathcal{A}_{G/e}, \tag{19}$$

so by Deletion-Restriction (Lemma 2.2) we have

$$\chi_{\mathcal{A}_G}(t) = \chi_{\mathcal{A}_{G-e}}(t) = \chi_{\mathcal{A}_{G/e}}(t).$$

To prove (19), define an affine isomorphism $\varphi : H_0 \xrightarrow{\cong} \mathbb{R}^{n-1}$ by

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, \ldots, x_i, \ldots, \hat{x_j}, \ldots, x_n),$$

where $\hat{x_j}$ denotes that the $j$th coordinate is omitted. (Hence the coordinates in $\mathbb{R}^{n-1}$ are $1, 2, \ldots, \hat{j}, \ldots, n$.) Write $H_{ab}$ for the hyperplane $x_a = x_b$ of $\mathcal{A}$. If neither of $a, b$ are equal to $i$ or $j$, then $\varphi(H_{ab} \cap H_0)$ is the hyperplane $x_a = x_b$ in $\mathbb{R}^{n-1}$. If $a \neq i, j$ then $\varphi(H_{ia} \cap H_0) = \varphi(H_{aj} \cap H_0)$, the hyperplane $x_a = x_i$ in $\mathbb{R}^{n-1}$. Hence $\varphi$

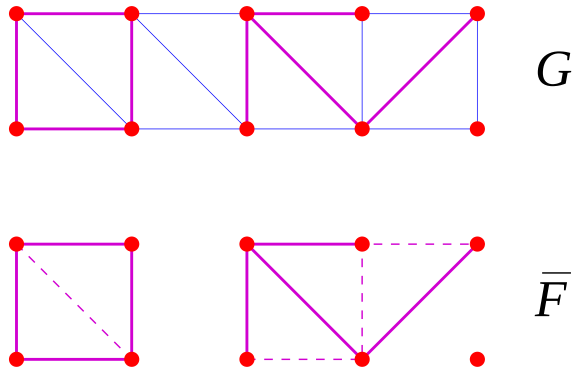



**Figure 6.** A graph $G$ with edge subset $F$ and closure $\bar{F}$

defines an isomorphism between $\mathcal{A}^{H_0}$ and the arrangement $\mathcal{A}_{G/e}$ in $\mathbb{R}^{n-1}$, proving (19).

Let $n\bullet$ denote the graph with $n$ vertices and no edges, and let $\emptyset$ denote the empty arrangement in $\mathbb{R}^n$. The theorem will be proved by induction (using Lemma 2.2) if we show:

(a) Initialization: $\chi_{n\bullet}(t) = \chi_{\emptyset}(t)$
(b) Deletion-contraction:

$$\chi_G(t) = \chi_{G-e}(t) - \chi_{G/e}(t) \tag{20}$$

To prove (a), note that both sides are equal to $t^n$. To prove (b), observe that $\chi_{G-e}(q)$ is the number of colorings of $\kappa : [n] \to [q]$ that are proper except possibly $\kappa(i) = \kappa(j)$, while $\chi_{G/e}(q)$ is the number of colorings $\kappa : [n] \to [q]$ of $G$ that are proper except that $\kappa(i) = \kappa(j)$. $\square$

Our second proof of Theorem 2.7 is based on Möbius inversion. We first obtain a combinatorial description of the intersection lattice $L(\mathcal{A}_G)$. Let $H_{ij}$ denote the hyperplane $x_i = x_j$ as above, and let $F \subseteq E(G)$. Consider the element $X = \bigcap_{ij \in F} H_{ij}$ of $L(\mathcal{A}_G)$. Thus

$$(x_1, \ldots, x_n) \in X \Leftrightarrow x_i = x_j \text{ whenever } ij \in F.$$

Let $C_1, \ldots, C_k$ be the connected components of the spanning subgraph $G_F$ of $G$ with edge set $F$. (A subgraph of $G$ is *spanning* if it contains all the vertices of $G$. Thus if the edges of $F$ do not span all of $G$, we need to include all remaining vertices as isolated vertices of $G_F$.) If $i, j$ are vertices of some $C_m$, then there is a path from $i$ to $j$ whose edges all belong to $F$. Hence $x_i = x_j$ for all $(x_1, \ldots, x_n) \in X$. On the other hand, if $i$ and $j$ belong to different $C_m$'s, then there is no such path. Let

$$\bar{F} = \{e = ij \in E(G) \,:\, i, j \in V(C_m) \text{ for some } m\},$$

where $V(C_m)$ denotes the vertex set of $C_m$. Figure 6 illustrates a graph $G$ with a set $F$ of edges indicated by thickening. The set $\bar{F}$ is shown below $G$, with the additional edges $\bar{F} - F$ not in $F$ drawn as dashed lines.

A *partition* $\pi$ of a finite set $S$ is a collection $\{B_1, \ldots, B_k\}$ of subsets of $S$, called *blocks*, that are nonempty, pairwise disjoint, and whose union is $S$. The set of all partitions of $S$ is denoted $\Pi_S$, and when $S = [n]$ we write simply $\Pi_n$ for $\Pi_{[n]}$. It follows from the above discussion that the elements $X_\pi$ of $L(\mathcal{A}_G)$ correspond to the

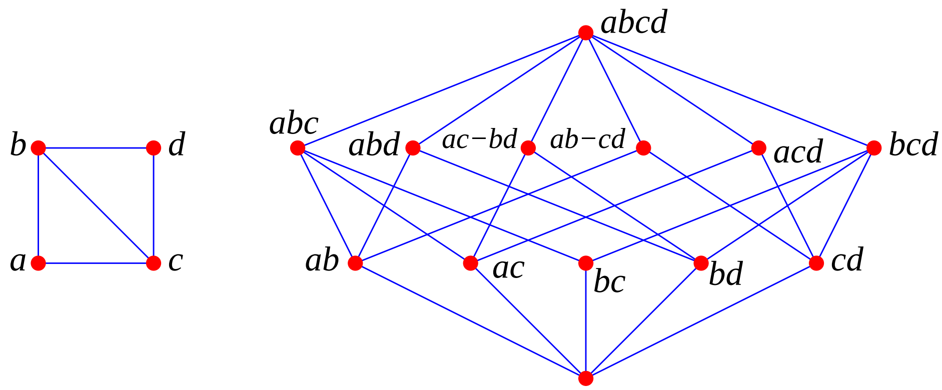



**Figure 7.** A graph $G$ and its bond lattice $L_G$

*connected partitions* of $V(G)$, i.e., the partitions $\pi = \{B_1, \dots, B_k\}$ of $V(G) = [n]$ such that the restriction of $G$ to each block $B_i$ is connected. Namely,

$$X_\pi = \{(x_1, \dots, x_n) \in K^n \, : \, i, j \in B_m \text{ for some } m \Rightarrow x_i = x_j\}.$$

We have $X_\pi \le X_\sigma$ in $L(\mathcal{A})$ if and only if every block of $\pi$ is contained in a block of $\sigma$. In other words, $\pi$ is a *refinement* of $\sigma$. This refinement order is the “standard” ordering on $\Pi_n$, so $L(\mathcal{A}_G)$ is isomorphic to an induced subposet $L_G$ of $\Pi_n$, called the *bond lattice* or *lattice of contractions* of $G$. (“Induced” means that if $\pi \le \sigma$ in $\Pi_n$ and $\pi, \sigma \in L(\mathcal{A}_G)$, then $\pi \le \sigma$ in $L(\mathcal{A}_G)$.) In particular, $\Pi_n \cong L(\mathcal{A}_{K_n})$. Note that in general $L_G$ is not a sublattice of $\Pi_n$, but only a sub-join-semilattice of $\Pi_n$ [why?]. The bottom element $\hat{0}$ of $L_G$ is the partition of $[n]$ into $n$ one-element blocks, while the top element $\hat{1}$ is the partition into one block. The case $G = K_n$ shows that the intersection lattice $L(\mathcal{B}_n)$ of the braid arrangement $\mathcal{B}_n$ is isomorphic to the full partition lattice $\Pi_n$. Figure 7 shows a graph $G$ and its bond lattice $L_G$ (singleton blocks are omitted from the labels of the elements of $L_G$).

**Second proof of Theorem 2.7.** Let $\pi \in L_G$. For $q \in \mathbb{P}$ define $\chi_\pi(q)$ to be the number of colorings $\kappa : [n] \to [q]$ of $G$ satisfying:

- If $i, j$ are in the same block of $\pi$, then $\kappa(i) = \kappa(j)$.
- If $i, j$ are in different blocks of $\pi$ and $ij \in E(G)$, then $\kappa(i) \ne \kappa(j)$.

Given *any* $\kappa : [n] \to [q]$, there is a unique $\sigma \in L_G$ such that $\kappa$ is enumerated by $\chi_\sigma(q)$. Moreover, $\kappa$ will be constant on the blocks of some $\pi \in L_G$ if and only if $\sigma \ge \pi$ in $L_G$. Hence

$$q^{|\pi|} = \sum_{\sigma \ge \pi} \chi_\sigma(q) \quad \forall \pi \in L_G,$$

where $|\pi|$ denotes the number of blocks of $\pi$. By Möbius inversion,

$$\chi_\pi(q) = \sum_{\sigma \ge \pi} q^{|\sigma|} \mu(\pi, \sigma),$$

where $\mu$ denotes the Möbius function of $L_G$. Let $\pi = \hat{0}$. We get

$$\chi_G(q) = \chi_{\hat{0}}(q) = \sum_{\sigma \in L_G} \mu(\sigma) q^{|\sigma|}. \tag{21}$$

It is easily seen that $|\sigma| = \dim X_\sigma$, so comparing equation (21) with Definition 1.3 shows that $\chi_G(t) = \chi_{\mathcal{A}_G}(t)$. $\square$

**Corollary 2.2.** *The characteristic polynomial of the braid arrangement $\mathcal{B}_n$ is given by*

$$\chi_{\mathcal{B}_n}(t) = t(t-1)\cdots(t-n+1).$$

**Proof.** Since $\mathcal{B}_n = \mathcal{A}_{K_n}$ (the graphical arrangement of the complete graph $K_n$), we have from Theorem 2.7 that $\chi_{\mathcal{B}_n}(t) = \chi_{K_n}(t)$. The proof follows from equation (17). $\square$

There is a further invariant of a graph $G$ that is closely connected with the graphical arrangement $\mathcal{A}_G$.

**Definition 2.7.** An *orientation* $\mathfrak{o}$ of a graph $G$ is an assignment of a direction $i \to j$ or $j \to i$ to each edge $ij$ of $G$. A *directed cycle* of $\mathfrak{o}$ is a sequence of vertices $i_0, i_1, \ldots, i_k$ of $G$ such that $i_0 \to i_1 \to i_2 \to \cdots \to i_k \to i_0$ in $\mathfrak{o}$. An orientation $\mathfrak{o}$ is *acyclic* if it contains no directed cycles.

A graph $G$ with no loops (edges from a vertex to itself) thus has $2^{\#E(G)}$ orientations. Let $R \in \mathcal{R}(\mathcal{A}_G)$, and let $(x_1, \ldots, x_n) \in R$. In choosing $R$, we have specified for all $ij \in E(G)$ whether $x_i < x_j$ or $x_i > x_j$. Indicate by an arrow $i \to j$ that $x_i < x_j$, and by $j \to i$ that $x_i > x_j$. In this way the region $R$ defines an orientation $\mathfrak{o}_R$ of $G$. Clearly if $R \neq R'$, then $\mathfrak{o}_R \neq \mathfrak{o}_{R'}$. Which orientations can arise in this way?

**Proposition 2.5.** *Let $\mathfrak{o}$ be an orientation of $G$. Then $\mathfrak{o} = \mathfrak{o}_R$ for some $R \in \mathcal{R}(\mathcal{A}_G)$ if and only if $\mathfrak{o}$ is acyclic.*

**Proof.** If $\mathfrak{o}_R$ had a cycle $i_1 \to i_2 \to \cdots \to i_k \to i_1$, then a point $(x_1, \ldots, x_n) \in R$ would satisfy $x_{i_1} < x_{i_2} < \cdots < x_{i_k} < x_{i_1}$, which is absurd. Hence $\mathfrak{o}_R$ is acyclic.

Conversely, let $\mathfrak{o}$ be an acyclic orientation of $G$. First note that $\mathfrak{o}$ must have a *sink*, i.e., a vertex with no arrows pointing out. To see this, walk along the edges of $\mathfrak{o}$ by starting at any vertex and following arrows. Since $\mathfrak{o}$ is acyclic, we can never return to a vertex so the process will end in a sink. Let $j_n$ be a sink vertex of $\mathfrak{o}$. When we remove $j_n$ from $\mathfrak{o}$ the remaining orientation is still acyclic, so it contains a sink $j_{n-1}$. Continuing in this manner, we obtain an ordering $j_1, j_2, \ldots, j_n$ of $[n]$ such that $j_i$ is a sink of the restriction of $\mathfrak{o}$ to $j_1, \ldots, j_i$. Hence if $x_1, \ldots, x_n \in \mathbb{R}$ satisfy $x_{j_1} < x_{j_2} < \cdots < x_{j_n}$ then the region $R \in \mathcal{R}(\mathcal{A})$ containing $(x_1, \ldots, x_n)$ satisfies $\mathfrak{o} = \mathfrak{o}_R$. $\square$

Note. The transitive, reflexive closure $\bar{\mathfrak{o}}$ of an acyclic orientation $\mathfrak{o}$ is a partial order. The construction of the ordering $j_1, j_2, \ldots, j_n$ above is equivalent to constructing a linear extension of $\mathfrak{o}$.

Let $\mathrm{AO}(G)$ denote the set of acyclic orientations of $G$. We have constructed a bijection between $\mathrm{AO}(G)$ and $\mathcal{R}(\mathcal{A}_G)$. Hence from Theorem 2.5 we conclude:

**Corollary 2.3.** *For any graph $G$ with $n$ vertices, we have $\#AO(G) = (-1)^n \chi_G(-1)$.*

Corollary 2.3 was first proved by Stanley in 1973 by a "direct" argument based on deletion-contraction (see Exercise 7). The proof we have just given based on arrangements is due to Greene and Zaslavsky in 1983.

Note. Given a graph $G$ on $n$ vertices, let $\mathcal{A}_G^{\#}$ be the arrangement defined by

$$x_i - x_j = a_{ij}, \quad ij \in E(G),$$

where the $a_{ij}$'s are generic. Just as we obtained equation (14) (the case $G = K_n$) we have

$$\chi_{\mathcal{A}_G^{\#}}(t) = \sum_F (-1)^{e(F)} t^{n-e(F)},$$

where $F$ ranges over all spanning forests of $G$.