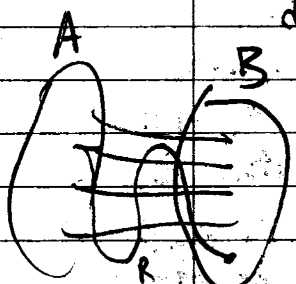
Pa K
11/2/05

Thm (Menger)
$G = (V,E)$ $A, B \subseteq V$ min # vertices separating A from B = max # nonint A-B paths in G

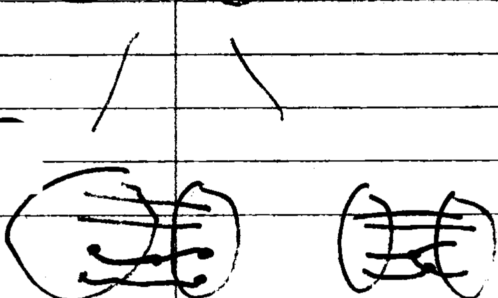
pf: Stronger Claim (this all comes from Diestel, 3.3 in book, diff't section online)

Here is a stronger claim: $\mathcal{P} = \{p_1, \dots p_\ell\}$ $\ell \le K \Rightarrow$ $\exists$ $\ell+1$ non int A-B paths in G whose endpts include those of p_i, where $\mathcal{P}$ is a set of non-intersecting A-B paths in G and K = max # of them

pf by induction on $|G-B|$. Given A, B, ℓ paths between, can find path to some vx in B that doesn't contain the start of any of the paths to some vx in A ~~s.t.~~ (possible since # endpts $\le$ connectivity). Call this path R. If R never intersects any of the other paths, we're done. O/w, let x be the last intersection of R + $\mathcal{P}$. Let $B' = B \cup Y \cup Z$ where Y = R from x to $b \in B$, Z = p_i from x to b, (i.e. tail of path that's crossed). Let $\mathcal{P}' = \mathcal{P} - p_i$ + beginning of p_i until x



So $\mathcal{P}'$ is a set of ℓ non-intersecting A - B' paths. $|G-B'| < |G-B|$ and $K' \ge K$ ~~since separ~~ (since $B' \supseteq B$). So by IH $\exists$ bigger set of paths containing those endpts. If new path's endpts are in A and $B \subseteq B'$, okay. O/w goes to Y or Z, ... ✓



Can think of this as generalization of Hall's. A, B parts of bip graph, then $\exists$ matching iff min # separating is $|A|$. If $\exists$ smaller sep'ng set, then... (to be talked about next time)

Thm (Gallai - Milgram) (2.3 in print, 2.5 online)
G digraph $\exists$ path cover $\mathcal{P} = \{p_1, \dots p_m\}$ and indep. set $\{v_1, \dots v_m\}$ s.t. $v_i \in P_i$
Where $\mathcal{P}$ is a set of non-intersecting paths containing all vertices. So min'l $\mathcal{P}$ $|\mathcal{P}| = m$
$\Rightarrow m \leq \alpha(G)$

Pf: Another one of those complicated induction proofs. Take $\mathcal{P}$ s.t. set of endpts $e(\mathcal{P})$ is min'l, i.e. $\nexists$ $\mathcal{P}'$ w/ $e(\mathcal{P}') \subsetneq e(\mathcal{P})$
Claim: $\exists$ $v_i \in P_i$ s.t. $\{v_i\}$ is indep.
Pf: Look at $p_1, \dots p_m$ w/ endpts $e_1, \dots e_m$.
If $(e_i e_j) \notin E$, then done. If $e_2 \to e_1$ and $|p_1| = 1$, then $\exists$ smaller set of paths $\leq$.
Let $G' = G - e_1$ $\mathcal{P}' = \{p_1 - e_1, p_2, \dots p_m\}$
Then $\mathcal{P}'$ is minimal, b/c can go from smaller set in G' to smaller in G
"Sorry for not writing this down, it's all written in Diestel"
min'l in G' $\Rightarrow \exists$ indep $\Rightarrow$ indep in G ✓