Pak

11/30/05

Solid partitions

SP = set of solid partitions $\lambda \subseteq \mathbb{Z}_+^3$ ($\lambda = \emptyset$ okay)

$|\lambda|$ = # cubes in λ

Thm (Percy MacMahon) MacMahon's f'la

$$\sum_{\lambda \in SP} t^{|\lambda|} = \prod_{i=1}^{\infty} \left(\frac{1}{1-t^i}\right)^i$$

Usual partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 0$

$\mathcal{P}$ = set of partitions

$\Rightarrow \sum_{\lambda \in \mathcal{P}} t^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1}{1-t^i}$ as he's sure we know

---

Conj

Let P_d = solid partitions in $\mathbb{Z}^d$

$$\sum_{\lambda \in P_d} t^{|\lambda|} = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^{\binom{n}{d-2}}}$$

is it right? Who knows?

can check by computer, false for $d = 4$

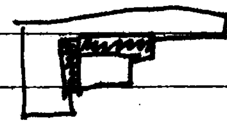
Def'n $f : \lambda \leftrightarrow [n]$ $|\lambda| = n$

standard Young tableau

SYT: put numbers from 1 to n in λ s.t. increase along rows L → R and on columns Top → Bottom

λ in this context is called a Young diagram

Def'n: $SYT(\lambda)$ = set of standard Young tableaux of shape λ



Thm (hook-length f'la)

$$|SYT(\lambda)| = \frac{n!}{\prod_{x \in \lambda} h_x}$$

where $x = (i,j) \Rightarrow$

$h_x = |\{i, \dots n\} \times \{j\} \cap \lambda| + |\{i\} \times \{j, \dots n\} \cap \lambda| - 1$

Def'n reverse plane partitions of shape λ $RPP(\lambda)$ is set of maps
$A : \lambda \to \mathbb{Z}$ s.t. $A(i,j) \leq A(i,j+1)$ and $A(i+1,j)$

$|A| = \sum_{(i,j)\in\lambda} A(i,j)$

Thm (hook-content f'la)

$$\sum_{A\in RPP(\lambda)} t^{|A|} = \prod_{x\in\lambda} 1/_{1-t^{h_x}}$$

Proposition: hook-content f'la $\Rightarrow$ MacMahon's f'la

Pf: $\lambda = (n,\dots,n) = [n]\times[n]$

$|RPP(\lambda)| = |\{SP \text{ which fit } n\times n \text{ footprint}\}|$

Thus $$\sum_{\lambda\in SP} t^{|\lambda|} = \lim_{n\to\infty} \underbrace{\sum_{A\in RPP(n^n)} t^{|A|}}_{n\times n \text{ square}} = \lim_{n\to\infty} \frac{1}{(1-t)(1-t^2)^2}\cdots\frac{1}{(1-t^n)^n}\cdot\left(\frac{1}{1-t^{n+1}}\right)^{n-1}\cdots\left(\frac{1}{1-t^{2n-1}}\right)$$

b/c of hooks in squares. Also, if you're looking at $n\to\infty$, coefficients are determined by just the first n factors (since eventually least power from 2nd half will be $n+1 >$ whatever)

So

$$\sum_{\lambda\in SP} t^{|\lambda|} = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^n} \quad ✓$$

Proposition: HCF $\Rightarrow$ HLF