11 Elasticity

What shape does a piece of paper take when we push it in at the ends? To answer this question let's acquaint ourselves with another continuum approximation, used to describe the deformation of elastic solids (we might actually have studied this before our work on fluids, as it is conceptually simpler). We first need to find a way to describe stress and strain within the solid, and then determine the relation between the two. Then we can derive the equations of elasticity and apply them to the buckling of a thin plate.

11.1 Strain

If a solid is deformed, then points within the solid will move. We fix our attention on a single point, whose coordinates are (x_1, x_2, x_3), and the close neighborhood of this point. We suppose that in the strained state the Cartesian coordinates of the same point have become (x'_1, x'_2, x'_3). The displacement of this point due to the deformation is denoted by $\boldsymbol{u} = \boldsymbol{u}(x_1, x_2, x_3)$, where

$$u_i = x'_i - x_i. \tag{259}$$

The vector $\boldsymbol{u}$ is called the *displacement vector*.

When a body is deformed the distance between its points change. Let's consider two points very close together. If the vector joining them before is dx_i, the vector joining them in the deformed body is $dx_i' = dx_i + du_i$. This distance between the points was originally $dl = \sqrt{dx_1^2 + dx_2^2 + dx_3^2}$ and is now $dl' = \sqrt{dx_1'^2 + dx_2'^2 + dx_3'^2}$. Using the summation convention, which tells us to sum over repeated indices (i.e., $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$), and substituting in that $du_i = (\partial u_i / \partial x_k) dx_k$, we get

$$dl'^2 = dl^2 + 2\frac{\partial u_i}{\partial x_k} dx_i dx_k + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l} dx_k dx_l. \tag{260}$$

We shall neglect the last of these terms, as we consider the u_i to be small, so that

$$dl'^2 = dl^2 + 2e_{ij} dx_i dx_j \tag{261}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{262}$$

are the components of the *strain tensor* $\boldsymbol{e}$. This is called linear elasticity (even though it is not really linear). It is often very useful to separate *pure shear* from *pure compression* effects, which can be achieved by rewriting[12]

$$e_{ij} = \left(e_{ij} - \frac{\delta_{ij}}{3} e_{ll}\right) + \frac{\delta_{ij}}{3} e_{ll} = \left(e_{ij} - \frac{\delta_{ij}}{3} \nabla \cdot \boldsymbol{u}\right) + \frac{\delta_{ij}}{3} \nabla \cdot \boldsymbol{u}. \tag{263}$$

The first part in parentheses has a vanishing trace and therefore represents pure shear.

11.2 Stress tensor

When a body is deformed, the arrangement of molecules within is changed, and forces arise that want to restore the body to its equilibrium configuration. These are called *internal stresses*, represented by a stress tensor $\sigma = (\sigma_{ij})$, and when there is no deformation the stress is zero

$$\sigma_{ik} = 0. \tag{264}$$

The three components of a force on a volume element V can be obtained from stresses by transforming a surface integral into a volume integral

$$\int_{\partial V} \sigma_{ik} dA_k = \int_V (\partial_k \sigma_{ik}) dV = \int_V f_i dV \tag{265}$$

Hence, the vector f_i must be given by the divergence of the *stress tensor* σ_{ik}.

$$f_i = \frac{\partial \sigma_{ik}}{\partial x_k}. \tag{266}$$

We recognize that $\sigma_{ik} dA_k$ is the force per unit area in the i-direction on the surface element with outward normal $d\boldsymbol{A}$. One thing we know about the stress tensor is that it is symmetric

[12] In d dimensions one would simply replace $\delta_{ij}/3$ by δ_{ij}/d everywhere.

($\sigma_{ij} = \sigma_{ji}$). If, for example, the body is in a gravitational field then the internal stresses must everywhere balance gravity, in which case the equilibrium equations are

$$\frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i = 0. \tag{267}$$

Additional external forces applied to the surface of the body will enters as boundary conditions that complement the equilibrium conditions (267). For instance, if there is an external force per unit area, $\hat{\boldsymbol{f}}$, acting over the surface, then we require

$$\sigma_{ik} n_k = \hat{f}_i, \tag{268}$$

where $\boldsymbol{n}$ is the outward unit normal on the surface.

11.3 Hooke's law

In general, we would like to use Eqs. (267) to predict the deformation of a solid body under a given force distribution. That is, we have to express the stress tensor σ_{ij} in terms of the displacement field $\boldsymbol{u}$. The main body of the mathematical theory of elasticity rests on the *assumption of a linear homogeneous relation between the elements of the stress tensor and the strain tensor*. This is just the continuum version of *Hooke's Law*. To simplify matters, let's focus on materials that are *isotropic* (i.e., the elastic properties are independent of direction). In this case

$$\sigma_{ij} = \lambda \delta_{ij}(e_{11} + e_{22} + e_{33}) + 2\mu e_{ij} = \lambda \delta_{ij} \mathrm{Tr}\boldsymbol{e} + 2\mu e_{ij} \tag{269}$$

where δ_{ij} is the Kronecker delta and λ and μ are positive elastic constants of the material, called *Lame coefficients*. The corresponding *(free) energy density* E of the body associated with deformation, obtained from the relation

$$\sigma_{ij} = \frac{\partial E}{\partial e_{ij}}, \tag{270}$$

is therefore

$$E = \frac{1}{2}\lambda e_{ii}^2 + \mu e_{ij}^2. \tag{271}$$

As stated above, the sum $e_{ii} = \mathrm{Tr}\boldsymbol{e}$ is related to the change in volume associated with a deformation. If this is zero, only the shape of the body is altered, corresponding to pure shear. Recalling our above decomposition

$$e_{ij} = (e_{ij} - \frac{1}{3}\delta_{ij} e_{ll}) + \frac{1}{3}\delta_{ij} e_{ll}. \tag{272}$$

we can obtain a general expression for the energy density of a deformed isotropic body, by replacing (271) with

$$E = \frac{1}{2} K e_{ll}^2 + \mu (e_{ik} - \frac{1}{3}\delta_{ik} e_{ll})^2 \tag{273}$$

where K and μ positive constants, respectively called the *modulus of compression* and the *modulus of rigidity*. In 3D, K is related to the Lame coefficients by[13]

$$K = \lambda + \frac{2}{3}\mu \tag{274}$$

[13] In 2D, this relation becomes $K = \lambda + \mu$.

11.4 A simple problem

Consider the simple case of a beam. Let the beam be along the z-axis, and let us pull it a both ends to stretch it. The force per unit area p is uniform over each end. The resulting deformation is uniform throughout the body and, hence, so is the stress tensor. It therefore follows that all components σ_{ik} are zero except for σ_{zz}, and from the forcing condition at the end we have that $\sigma_{zz} = p$.

From the general expression that relates the components of the stress and strain tensors, we see that all components e_{ik} with $i \neq k$ are zero. The equilibrium equations are therefore

$$e_{xx} = e_{yy} = -\frac{1}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right) p \tag{275}$$

and

$$e_{zz} = \frac{1}{3}\left(\frac{1}{\mu} + \frac{1}{3K}\right) p. \tag{276}$$

The component e_{zz} gives the lengthening of the rod, and the coefficient of p is the *coefficient of extension*. Its reciprocal is *Young's modulus*

$$Y = \frac{9K\mu}{3K + \mu}. \tag{277}$$

The components e_{xx} and e_{yy} give the relative compression of the rod in the transverse direction. The ratio of the transverse compression to the longitudinal extension is called *Poisson's ratio*, ν:

$$e_{xx} = -\nu e_{zz}, \tag{278}$$

where

$$\nu = \frac{1}{2}\left(\frac{3K - 2\mu}{3K + \mu}\right). \tag{279}$$

Since K and μ are always positive, Poisson's ratio can vary between -1 and $\frac{1}{2}$. Note that a negative value corresponds to pulling on the beam and it getting thicker! Now we see why we use Y and ν; they are easier to measure. Inverting these formulae, we get

$$\mu = \frac{Y}{2(1 + \nu)}, \qquad K = \frac{Y}{3(1 - 2\nu)}. \tag{280}$$

The free energy then becomes

$$E = \frac{Y}{2(1 + \nu)}\left(e_{ik}^2 + \frac{\nu}{1 - 2\nu} e_{ll}^2\right). \tag{281}$$

The stress tensor is given in terms of the strain tensor by

$$\sigma_{ik} = \frac{Y}{1 + \nu}\left(e_{ik} + \frac{\nu}{1 - 2\nu} e_{ll}\delta_{ik}\right). \tag{282}$$

Conversely

$$e_{ik} = \frac{1}{Y}[(1 + \nu)\sigma_{ik} - \nu\sigma_{ll}\delta_{ik}]. \tag{283}$$

11.5 Bending of a thin beam

Now we are in a position to try and calculate the shape of a bent beam. Our analysis requires that the thickness be much smaller than the lateral dimension. The deformations must also be small, such that the displacements are small compared with the thickness. Although the general equilibrium equations are greatly simplified when considering thin plates, it is more convenient not to derive our result from these. Rather, we shall use our knowledge of variational calculus to calculate afresh the energy of a bent plate, and set about varying that energy.

When a plate is bent, it is stretched at some points and compressed at others: on the convex side there is evidently an extension and on the concave side there is compression. Somewhere in the middle there is a *neutral surface*, on which there is no extension or compression. The neutral surface lies midway through the plate.

We take a coordinate system with the origin on the neutral surface and the z-axis normal to the surface. The xy-plane is that of the undeformed surface. The displacement of the neutral surface is given by $u_z = w(x, y)$. For further calculations we note that since the plate is thin, comparatively small forces on the surface are needed to bend it. These forces are always considerably less than the internal stresses caused in the deformed beam by the extension and compression of its parts. Thus we have on both surfaces of the plate

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0. \tag{284}$$

Since the plate is small, these quantities must be small within the plate if they are zero on the surface. We therefore conclude that $\sigma_{xz} = \sigma_{yz} = \sigma_{zz}$ are small everywhere, and equate them to zero. From our general formulae relating stress and strain, we have

$$\sigma_{zx} = \frac{Y}{1+\nu} e_{zx}, \qquad \sigma_{zy} = \frac{Y}{1+\nu} e_{zy}, \tag{285}$$

$$\sigma_{zz} = \frac{Y}{(1+\nu)(1-2\nu)}[(1-\nu)e_{zz} + \nu(e_{xx} + e_{yy})]. \tag{286}$$

Substituting in our expression for the the strain tensor and equating to zero, we get

$$\frac{\partial u_x}{\partial z} = -\frac{\partial u_z}{\partial x}, \qquad \frac{\partial u_y}{\partial z} = -\frac{\partial u_z}{\partial y}, \tag{287}$$

$$e_{zz} = -\frac{\nu(e_{xx} + e_{yy})}{(1-\nu)}. \tag{288}$$

In the first two of these equations u_z can be replaced by $w(x, y)$. Thus, integrating the above relations gives

$$u_x = -z\frac{\partial w}{\partial x}, \qquad u_y = -z\frac{\partial w}{\partial y}, \tag{289}$$

where the constants of integration were chosen so as to make $u_x = u_y = 0$ for $z = 0$. Knowing u_x and u_y we can now determine all the components of the strain tensor:

$$e_{xx} = -z\frac{\partial^2 w}{\partial x^2}, \qquad e_{yy} = -z\frac{\partial^2 w}{\partial y^2} \qquad e_{xy} = -z\frac{\partial^2 w}{\partial x \partial y}, \tag{290}$$

$$e_{xz} = e_{yz} = 0, \qquad e_{zz} = \frac{z\nu}{1-\nu}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right). \tag{291}$$

We now calculate the free energy of the plate, using our general formula,

$$E = z^2 \frac{Y}{1+\nu}\left[\frac{1}{2(1-\nu)}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]. \tag{292}$$

Integrating from $-\frac{h}{2}$ to $\frac{h}{2}$, where h is the thickness of the plate, then integrating again over an area element gives the free energy per unit area

$$E_A = \frac{Yh^3}{24(1-\nu^2)} \int\int \left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} dxdy. \tag{293}$$

where the element of area can be written $dxdy$ since the deformation is small.

We now derive the equilibrium equation for a plate from the condition that it's free energy is a minimum. To simplify things, let's just ignore any y-dependence and consider a 2D problem. Using the calculus of variations we have that the energy of the distorted beam is

$$\frac{\delta E}{\delta w} = \frac{Yh^3}{12(1-\nu^2)}\frac{d^4 w}{dx^4}. \tag{294}$$

This expression must equal the force $f(x)$ applied to deform the plate:

$$\frac{Yh^3}{12(1-\nu^2)}\frac{d^4 w}{dx^4} = f(x). \tag{295}$$

The simplest boundary conditions are if the edges are clamped, in which case

$$w = 0, \qquad \frac{dw}{dx} = 0 \tag{296}$$

at the edges. The first of these expresses the fact that the edge of the plate undergoes no deformation, and the second that it remains horizontal. For more details, see chapter 2 in *Theory of Elasticity, Landau & Lifschitz.*

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