11/22/04

Thue's Theorem.

$$ax^3 + by^3 = c \quad a, b, c \in \mathbb{Z} \text{ nonzero.}$$

finitely many integer solutions.

$\Longleftrightarrow$ DAT

$\beta = \sqrt[3]{b}$ $b \neq 0$ integer not a perfect cube.

Then there are finitely many p/q with

$$\left| \frac{p}{q} - \beta \right| \leq \frac{C}{q^3} \qquad C \geq 0.$$

Want to construct $F(X, Y)$ s.t.

$F(\beta, \beta) = 0$

F vanish to high order at (β, β).

Today: Find F.

<u>Siegel</u>'s <u>Lemma</u>: Let $N > M$ pos. integers. Given a homogenous linear System of M equations in N unknowns: $A\bar{t} = \bar{0}$

A = $M \times N$ matrix. $\bar{t}$ = $N \times 1$ $\bar{0}$ = $M \times 1$

with integer coefficients

Then there exists a nonzero integer solution

$\bar{t} = (t_1, \ldots, t_N)$ satisfying

$$\max_{1 \leq i \leq N} |t_i| < 2 \left(4N \max_{i,j} |A_{ij}| \right)^{\frac{M}{N-M}}$$

$$F(X, Y) = P(X) + Y Q(X)$$

$\deg(P(X)), \deg(Q(X)) \leq m + n$

Let n be a large positive integer.

Let $m = \lfloor 2n/3 \rfloor$

$(X-\beta)^n \mid F(X,Y)$

$$\frac{d^k}{dx^k}(x^n) = \frac{n!}{(n-k)!} x^{n-k}.$$

Notice $k! \,\Big|\, \frac{n!}{(n-k)!}$ since $\binom{n}{k}$ is an integer.

$$F^{(k)}(X,Y) \equiv \frac{1}{k!}\frac{\partial^k}{\partial x^k} F(X,Y) = \frac{1}{k!}\left(\frac{d^k}{dx^k}P(x) + \frac{d^k}{dx^k}Q(x)\,Y\right)$$

If F has integer coefficients, so does $F^{(k)}$.

$(X-\beta)^n \mid F(X,Y)$ iff $F^k(\beta,\beta)=0$ $\quad 0 \le k < n.$

Write

$$P(x) = \sum_{i=0}^{m+n} u_i X^i \qquad Q(x) = \sum_{i=0}^{m+n} v_i X^i$$

$$F^k(\beta,\beta) = \sum_{i=k}^{m+n} \binom{i}{k}\left(u_i \beta^{i-k} + v_i \beta^{i-k+1}\right)$$

$$= \sum_{i=0}^{m+n-k} \binom{i+k}{k} \beta^i u_{i+k} + \sum_{i=0}^{m+n-k+1} \binom{i+k-1}{k} \beta^i v_{i+k-1}$$

$$\sum_{i=0}^{m+n-k+1} \left(\binom{i+k}{k}\beta^i u_{i+k} + \binom{i+k-1}{k}\beta^i v_{i+k-1}\right)$$

Note: $u_i, v_i = 0$ if $i < 0$ or $i > m+n$.

Want: $F^k(\beta,\beta) = 0$ for $0 \le k < n$.

$4 \cdot 2(m+n+1) \leq 2^{m+n+3} \leq 4^{m+n}$
(Assume $m \geq 3$)

This gives us $\max\{|u_i|, |v_i|\} \leq 2\left((16b)^{m+n}\right)^{\frac{3n}{2(m+n+1)-3n}}$

$$\frac{3n}{2(m+n+1)-3n} = \frac{3}{2\frac{m+1}{n} - 1} \leq 9.$$

Auxiliary Polynomial Theorem

β as before, b as before. and m, n integers satisfying $m+1 \geq \frac{2n}{3} \geq m \geq 3$.

Then $\exists\ F(X, Y) = P(X) + Y\,Q(X)$ as before.

$$\max_{0 \leq i \leq m+n} \{|u_i|, |v_i|\} \leq 2 \cdot (16b)^{9(m+n)}$$

Example on p. 164 of book $n=5, m=3$ $\beta = \sqrt[3]{2}$
$3n = 15$ equations in $2(m+n+1) = 18$ unknowns. and
Look at p. 164 for the matrix entries.

One of the polynomials they get is
$-8 - 64x^3 - 20x^6 + Y(40x^2 + 32x^5 + x^8)$

$\frac{29}{23} = 1.2608\ldots$ $\quad \frac{635}{504} = 1.2599206\ldots$

$\sqrt[3]{2} \approx 1.2599210\ldots$

$F\left(\frac{29}{23}, \frac{635}{504}\right) \approx -.000714$

$\beta^3 = b$ is an integer. so $\beta^{3j+\ell} = b^j \beta^\ell$. $\ell = 0,1,2$

$$F^k(\beta,\beta) = \sum_{\ell=0}^{2} \Big(\sum_j \binom{3j+\ell+k}{k} b^j u_{3j+\ell+k}$$

$$+ \binom{3j+\ell+k-1}{k} b^j v_{3j+\ell+k-1} \Big)\beta^\ell$$

β has minimal poly. $x^3 - b$

So $a + c\beta + d\beta^2 = 0$, $a, c, d \in \mathbb{Q} \Rightarrow a = c = d = 0$.

So we want

$$0 = \sum_j \left(\binom{3j+\ell+k}{k} b^j u_{3j+\ell+k} + \binom{3j+\ell+k-1}{k} b^j v_{3j+\ell-k-1} \right)$$

for every $\ell \in \{0,1,2\}$ and $k \in \{0,1,\dots,n-1\}$.

We have $3n$ equations (homogeneous linear) equations in $2(n+m+1)$ unknowns. u_i, v_i with integer coefficients.

Seigel says: there is an integer solution s.t.

$$\max_{0 \le i \le m+n} \{|u_i|, |v_i|\} \le 2\big(4 \cdot 2(m+n+1)\mu\big)^{\frac{3n}{2(n+m+1)-3n}}$$

Want to estimate μ. $\binom{N}{M} \le 2^N$

$$\max_{\substack{j,k,\ell \\ 0 \le 3j+\ell \le m+n \\ 0 \le k < n}} \binom{3j+\ell+k}{k} b^j \le \max_{\substack{0 \le i \le m+n \\ 0 \le k < n}} 2^{i+k} b^{i/3}$$

$$= 2^{m+2n-1} \cdot b^{m+n/3} \le (4b)^{m+n}$$