Recall. If $\Sigma_1 \subset k^{n_1}$ and $\Sigma_2 \subset k^{n_2}$ are algebraic subsets, then a *morphism* $\rho : \Sigma_1 \to \Sigma_2$ is a set map such that there exist polynomials $f_1, \ldots, f_{n_2} \in k[x_1, \ldots, x_{n_1}]$ so that $\rho(\underline{a}) = (f_1(\underline{a}, \ldots, f_{n_2}(\underline{a}))$.

Thm. 9 There is a natural bijection $Hom(\Sigma_1, \Sigma_2) \cong Hom_{k-Alg}(R_2, R_1)$ where $R_i = k[x_1, \ldots, x_{n_i}]/I(\Sigma_i)$.

Proof. Define $\alpha : \mathrm{Hom}(\Sigma_1, \Sigma_2) \to \mathrm{Hom}(R_2, R_1)$. Given ρ, choose $f_1, \ldots, f_{n_2}$ inducing ρ and use the map τ between $k[x_1, \ldots, x_{n_2}]$ to $k[x_1, \ldots, x_{n_1}]$ defined by $x_i \mapsto f_i$.

Claim. This extends naturally to a map between R_2 and R_1. To construct this, we wish to find ρ^* such that ρ^* commutes with τ and the surjective maps onto R_2 and R_1. In order to prove that ρ^* exists, we need that if $g(x_1, \ldots, x_{n_2}) \in I(\Sigma_2)$ then $g(f_1(\underline{x}), \ldots, f_{n_2}(\underline{x})) \in I(\Sigma_1)$. Say that $\underline{a} \in \Sigma_1$. Then this is $g(\rho(\underline{a})) = 0$.

We also wish to show that ρ^* is unique, independent of the choices that induce τ. Thus, suppose τ' is another such map, defined by a second set of choices, $f'_1, \ldots, f'_{n_2}$. Then we need to prove that $\tau(x_i) - \tau'(x_i) \in I(\Sigma_1)$. (This will show that the map is natural.) All we need to do is note that $f'_i = f_i + g_i$ for some $g_i \in I(\Sigma_1)$, and we are done. (?)

This all shows that α is well defined: it maps ρ to ρ^*. To show that α is injective, we must show how to construct ρ from $\rho^* = \alpha(\rho)$. $\rho : \Sigma_1 \to \Sigma_2$ is given by $m \subset R_1 \mapsto \rho^{*-1}(m) \subset R_2$ (this is a map on maximal ideals). This works.

To show that α is surjective, we consider a given ρ^* in $\mathrm{Hom}(R_2, R_1)$ and we will try to lift this to a $\rho \in \mathrm{Hom}(\Sigma_1, \Sigma_2)$. So to do this, we use a commutative diagram, by lifting ρ^* to τ and then to a map from Σ_1 to Σ_2 defined by $(a_1, \ldots, a_{n_1}) \mapsto (\tau(x_1)(\underline{a}), \ldots, \tau(x_{n_2})(\underline{a}))$.

Martin says, study this until it's very clear. This is really a diagram argument.

Example:

Σ_1 is the parabola $y = x^2$ and Σ_2 is the line $y = 0$. Then, the only natural morphism takes x to x, while taking 0 to x^2. This produces a map from $k[x, y]/(y)$ to $k[x, y]/(y - x^2)$.

Example: $k = \mathbb{A} \to C = \{(a, a^2, a^3 | a \in k\} \subset \mathbb{A}^3$. Then, we get a map
$k[X, Y, Z]/(Y - X^2, Z - X^3) \to k[T]$ defined by $X \mapsto T, Y \mapsto T^2, Z \mapsto T^3$. The inverse map is $T \mapsto X$.

Def. If $\Sigma \subset k^n$ is an alg. set, then the *affine coordinate ring* of Σ is $\Gamma(\Sigma) = k[x_1, \ldots, x_n]/I(\Sigma)$. This will, in a sense, be independent of the embedding of Σ in k^n.

Lemma There is a natural bijection $\Gamma\Sigma \cong \mathrm{Hom}(\Sigma, \mathbb{A}')$.

Pf. $\mathrm{Hom}(\Sigma, \mathbb{A}') \cong \mathrm{Hom}(k[T], \Gamma(\Sigma)) \cong \Gamma(\Sigma)$, where this last is defined by $\rho^* \mapsto \rho^*(T)$. The first part is given by our previous theorem.

Goal: Γ induces an equivalence of categories between the category of irreducible affine algebraic sets and the category of integral k-algebras of finite type.

Now, a brief divergence into category theory.

Def. A category $\mathcal{C}$ is a set of "objects" $Ob(\mathcal{C})$. For any two objects $X, Y \in Ob(\mathcal{C})$ there is a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ called the set of morphisms from X to Y. Further, for $X, Y, Z \in Ob(\mathcal{C})$, a composition map $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is defined such that

1. $\forall X \in Ob(\mathcal{C})$ there is an $id_X \in \mathrm{Hom}(X,X)$ which is a right (left) identity for elements in $\mathrm{Hom}(Y,X)$ ($\mathrm{Hom}(X,Y)$).
2. Composition is associative.

Example. The categories of affine algebraic sets, modules over a given ring, and k-algebras are sensible categories. The maps are defined just as they should be.

Def. If $\mathcal{C}$ and $\mathcal{D}$ are categories, a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is an association where for each $A \in Ob(\mathcal{C})$ we get a particular $F(A) \in Ob(\mathcal{D})$. Also, for all $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ we get a map $F(f) : F(Y) \to F(X)$, preserving composition and identity.

A covariant functor gives, from f, a map $F(f) : F(X) \to F(Y)$.

To return to our goal, the functor that takes Σ to $\Gamma(\Sigma)$ and that takes morphisms to k-algebra maps is a covariant functor. We have already proved all the parts we need for this.

Def. F is an equivalence if there is an "inverse" functor F^{-1} such that for every object $A \in \mathcal{D}$ is isomorphic to $F(X)$ for some $X \in \mathcal{C}$ and this map is F^{-1}. Furthermore, for all $X, Y \in \mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$ is also bijective.

To show that Γ is an equivalence, we need to note that both the maps of morphisms and the maps of objects are bijective. We have already shown this for the morphisms. Suppose R is an integral k-algebra of finite type, meaning $R = k[x_1, \ldots, x_n]/P$ for some prime ideal P.

Remark: this equivalence fails for algebraic sets in $\mathbb{P}^n$.