18.781 Problem Set 3

Thursday, March 1.

Collaboration is allowed and encouraged. However, your writeups should be your own, and you must note on the front the names of the students you worked with.

Extensions will only be given for extenuating circumstances.

1. Solve (by hand) the congruence $x^3 - 9x^2 + 23x - 15 \equiv 0 \pmod{143}$.

2. What are the last two digits of 2^{100} and of 3^{100}? (Don't use a computer for this, either!)

3. Find the number of solutions of $x^2 \equiv x \pmod{m}$ for any positive integer m.

4. (a) Show that the number $n = 561 = 3 \cdot 11 \cdot 17$ satisfies the property P: for any a coprime to n, we have $a^{n-1} \equiv 1 \pmod{n}$.
   (b) Let n be a squarefree composite number satisfying P. Show that n has at least 3 prime factors.
   (c) Write down a sufficient condition for $n = pqr$ (where p, q, r are primes) to satisfy property P. Then write a gp program to generate a list of ten such numbers n.

5. Do there exist arbitrarily long sequences of consecutive integers, none of which are squarefree? (i.e. given any positive integer N, does there exist a sequence of integers $x, x+1, \ldots, x+N-1$ such that none of these is squarefree?) Prove your assertion.

6. This computational exercise will involve the notion of "density". We say that a set S of primes has density δ if the limit

$$\lim_{N\to\infty} \frac{\#\{p < N : p \text{ prime and } p \in S\}}{\#\{p < N : p \text{ prime }\}}$$

   exists and equals δ.

   (a) Let $f(x) = x^3 - 2$. Write a gp program to calculate the set S of primes p less than 10000 such that f has a solution modulo p. Make a conjecture about the density of such primes.
   (b) Now do the same exercise for $f(x) = x^3 - 3x - 1$.
   (c) (Bonus) What qualitative feature of f differentiates these two cases?