18.905 Problem Set 7

Due Friday, October 27 in class

The method of acyclic models.

5 questions. Do any 4 for full credit; all 5 for bonus marks.

Suppose that $\mathcal{C}$ is a category, and F is a functor from $\mathcal{C}$ to abelian groups. Suppose that for each $i \in I$, M_i is an object of $\mathcal{C}$ and $x_i \in F(M_i)$. We say that F is *free with models* $\{(M_i, x_i)\}_{i \in I}$ if, for all $c \in \mathcal{C}$, $F(c)$ is free with basis

$$\{F(\sigma)(x_i) | i \in I, \sigma : M_i \to c\}.$$

In other words, we can write any element of $F(c)$ uniquely as a sum with finitely many nonzero terms

$$\sum_i \sum_{\sigma : M_i \to c} n_{i,\sigma} F(\sigma)(x_i).$$

We will refer to $\{M_i\}$ as the set of models of F.

Let Top^2 be the category of pairs (X, Y) of topological spaces, where a map $(X, Y) \to (X', Y')$ is a pair of maps (f, g), where $f : X \to X'$ and $g : Y \to Y'$.

1. Define functors from Top^2 to abelian groups by

$$A_n(X, Y) = C_n(X \times Y)$$

$$B_n(X, Y) = \bigoplus_{p+q=n} C_p(X) \otimes C_q(Y)$$

   Show that A_n is free with a single model (Δ^n, Δ^n), and B_n is free with set of models $\{(\Delta^p, \Delta^q) \mid p + q = n\}$. Show that $A_0(X, Y)$ and $B_0(X, Y)$ are isomorphic in a natural way, and that this isomorphism preserves elements in the image of the boundary map.

2. The functors $A_n(X, Y)$ and $B_n(X, Y)$ assemble into the chain complexes $A_*(X, Y) = C_*(X \times Y)$ and $B_*(X, Y) = C_*(X) \otimes C_*(Y)$. Use the algebraic Künneth formula, together with facts we already know, to show that

$$H_k(A_*(\Delta^p, \Delta^q)) = H_k(B_*(\Delta^n, \Delta^n)) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k > 0. \end{cases}$$

   (We say that the model (Δ^n, Δ^n) is *acyclic* for B_*, and the models (Δ^p, Δ^q) are *acyclic* for A_*.) Conclude that the chain complexes $A_*(\Delta^p, \Delta^q)$ and $B_*(\Delta^n, \Delta^n)$ are exact except in dimension 0.

3. If a functor F is free with models $\{(M_i, x_i)\}_{i \in I}$, give a brief explanation why (similarly to a problem on a previous assignment) a natural transformation Θ from F to another functor G is the same as a choice of $\Theta_{M_i}(x_i)$ for all i.

4. Suppose that we have functors F_*, G_* from a category $\mathcal{C}$ to the category of chain complexes, and natural transformations $\Theta_n : F_n \to G_n$ for $n < N$ such that $\Theta_{n-1} \circ \partial = \partial \circ \Theta_n$ whenever both sides are defined.

   Suppose that F_N is free with models $\{(M_i, x_i)\}_{i \in I}$, and for all i the sequence
   $$G_N(M_i) \to G_{N-1}(M_i) \to G_{N-2}(M_i)$$
   is exact, or such that Θ_{N-1} carries boundaries to boundaries.

   Show that we can find a natural transformation $\Theta_N : F_N \to G_N$ such that $\Theta_{N-1} \circ \partial = \partial \circ \Theta_N$. Hint: Find elements $y_i \in G_N(M_i)$ such that $\partial y_i = \Theta_{N-1}(\partial x_i)$, and apply problem 3.

   Conclude that there are natural chain maps $A_*(X, Y) \to B_*(X, Y)$ and $B_*(X, Y) \to A_*(X, Y)$ extending the isomorphism in dimension zero.

5. Suppose that we have functors F_*, G_* from a category $\mathcal{C}$ to the category of chain complexes, and *two* sets of natural transformations $\Theta_n, \Sigma_n : F_n \to G_n$ such that $\Theta_{n-1} \circ \partial = \partial \circ \Theta_n$ and $\Sigma_{n-1} \circ \partial = \partial \circ \Sigma_n$.

   Assume that we have natural transformations $H_n : F_n \to G_{n+1}$ for $n < N$ such that $\partial \circ H_n + H_{n-1} \circ \partial = \Theta_n - \Sigma_n$.

   Suppose that F_N is free with models $\{(M_i, x_i)\}_{i \in I}$, and for all i the sequence
   $$G_{N+1}(M_i) \to G_N(M_i) \to G_{N-1}(M_i)$$
   is exact.

   Show that we can find a natural transformation $H_N : F_N \to G_{N+1}$ such that $\partial \circ H_N + H_{N-1} \circ \partial = \Theta_N - \Sigma_N$. Hint: Find elements $y_i \in G_{N+1}(M_i)$ such that $\partial y_i = \Theta_N(x_i) - \Sigma_N(x_i) - H_{N-1}(\partial x_i)$, and apply problem 3.

   Conclude that any two natural chain maps $A_*(X, Y)$ and $B_*(X, Y)$ extending the isomorphisms in degree zero are chain homotopic, hence induce the same map on homology, and that any composite natural chain maps
   $$A_*(X, Y) \to B_*(X, Y) \to A_*(X, Y)$$
   and
   $$B_*(X, Y) \to A_*(X, Y) \to B_*(X, Y)$$
   induce the identity map on homology.