LECTURE 15: EILENBERG-MACLANE SPACES

1. Construction

Let π be a group. Recall, a pointed CW complex X is a $K(\pi, n)$ (Eilenberg-MacLane space) if

$$\pi_k(X) = \begin{cases} \pi & k = n \\ 0 & k \neq n \end{cases}$$

- A $K(\pi, 0)$ is a CW complex with $\pi_0 = \pi$ having contractible components. For example, π is a $K(\pi, 0)$ when viewed as a discrete space.
- π must be abelian for $n > 1$.

Let n be greater than or equal to 1, and let π be abelian. A pointed CW complex X is a $M(\pi, n)$ (Moore space) if it is $(n-1)$-connected and has

$$\widetilde{H}_k(X) = \begin{cases} \pi & k = n \\ 0 & k \neq n \end{cases}$$

Construction of Moore spaces. We will assume for simplicity that $n \geq 2$. Form a short exact sequence of abelian groups.

$$0 \to \bigoplus_J \mathbb{Z} \xrightarrow{\alpha} \bigoplus_I \mathbb{Z} \to \pi \to 0$$

The Hurewicz homomorphism and the universal property of the wedge prove the following lemma.

Lemma 1.1. Assume $n \geq 2$. There is a canonical isomorphism

$$[\bigvee_J S^n, \bigvee_I S^n]_* \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\{J\}, \mathbb{Z}\{I\})$$

where $\mathbb{Z}\{S\}$ denotes the free abelian group generated by a set S.

There therefore exists a map

$$\widetilde{\alpha} : \bigvee_I S^n \to \bigvee_J S^n$$

which induces α on $\widetilde{H}_n$. Define $M(\pi, n)$ to be the cofiber $C(\alpha)$. The long exact sequence of a cofiber implies that the homology of $M(\pi, n)$ has the desired properties.

Construction of Eilenberg-MacLane spaces. For simplicity, continue to assume that $n \geq 2$. The arguments have to be slightly altered for $n = 1$. We construct $K(\pi, n)$ as

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a colimit $\varinjlim K^t$ where we inductively kill the higher homotopy groups of $M(\pi, n)$. Inductively construct cofibers:

$$\begin{tikzcd}
\bigvee_{\pi_{n+1}(K^0)} S^{n+1} \arrow[r] & K^0 \arrow[r, equal] \arrow[d] & M(\pi, n) \\
\bigvee_{\pi_{n+2}(K^1)} S^{n+2} \arrow[r] & K^1 \arrow[d] & \\
\bigvee_{\pi_{n+3}(K^2)} S^{n+3} \arrow[r] & K^2 \arrow[d] & \\
& \vdots &
\end{tikzcd}$$

The properties of the cofiber imply that if X is another Eilenberg-MacLane space of type (π, n), then there exists a weak equivalence

$$K(\pi, n) \xrightarrow{\simeq} X.$$

Since X is a CW complex, we deduce that this must be a homotopy equivalence. We therefore have the following proposition.

Proposition 1.2. Any two Eilenberg-MacLane spaces of type (π, n) are homotopy equivalent.

Corollary 1.3. There are homotopy equivalences

$$K(\pi, n) \xrightarrow{\simeq} \Omega K(\pi, n+1).$$

In particular, $K(\pi, n)$'s are infinite loop spaces.

Corollary 1.4. A $K(\pi, n)$ is a homotopy commutative H-space.

2. Representing cohomology

We wish to prove the following theorem.

Theorem 2.1. There is a natural transformation of homotopy invariant functors $\mathrm{Top}_* \to \mathrm{AbelianGroups}$

$$[-, K(\pi, n)]_* \to \widetilde{H}^n(-, \pi).$$

This natural transformation is an isomorphism when restricted to the subcategory of pointed CW complexes.

In this lecture, we merely construct the natural transformation. This is obtained by producing a "fundamental class"

$$[\iota_n] \in \widetilde{H}^n(K(\pi, n), \pi).$$

This should be thought of as the universal cohomology class. This trick is often used when dealing with representable functors. It induces a transformation (natural in X)

$$\eta_X : [X, K(\pi, n)]_* \to \widetilde{H}^n(X, \pi)$$

by $\eta_X[f] = f^*[\iota_n]$.

The fundamental class is obtained by the universal coefficient theorem. There is a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\widetilde{H}_{n-1}(K(\pi,n)),\pi) \to \widetilde{H}^n(K(\pi,n),\pi) \to \mathrm{Hom}_{\mathbb{Z}}(\widetilde{H}_n(K(\pi,n)),\pi) \to 0$$

which, using the Hurewicz theorem, gives an isomorphism

$$\widetilde{H}^n(K(\pi,n),\pi) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\pi,\pi).$$

The fundamental class $[\iota_n]$ corresponds to the identity map $Id : \pi \to \pi$.