LECTURE 33: CHERN CLASSES AND ELEMENTARY SYMMETRIC POLYNOMIALS

Today we describe how the sum formula and the splitting principle are related to the theory of symmetric polynomials.

We proved the splitting principle inductively by splitting off one line at a time:

$$L_1 \oplus \cdots \oplus L_n \longrightarrow \cdots \longrightarrow L_2 \oplus L_1 \oplus V_2 \longrightarrow L_1 \oplus V_1 \longrightarrow V$$
$$P(V_{n-2}) \longrightarrow \cdots \longrightarrow P(V_1) \longrightarrow P(V) \longrightarrow X$$

Here, the maps on the base spaces all induce injections on cohomology.

We wish to implement this for the universal vector bundle $V_{univ}^n \to BU(n)$.

We first record a general lemma

Lemma 0.1. There is an equivalence

$$B(G \times H) \simeq BG \times BH.$$

Proof. One just needs to check that $G \times H$-bundles are products of H-bundles and G-bundles. □

Observe that we have a fibration

$$U(n)/(U(1) \times U(n-1)) \to B(U(1) \times U(n-1)) \to BU(n).$$

The universal bundle pulls back to $L_{univ} \boxplus V_{univ}^{n-1}$. The action of $U(n)$ on $\mathbb{C}P^{n-1}$ has stabilizer $U(1) \times U(n-1)$, giving an identification of the fiber as

$$U(n)/(U(1) \times U(n-1)) = \mathbb{C}P^{n-1}.$$

In fact, identifying $B(U(1) \times U(n-1))$ with $EU(n)/(U(1) \times U(n-1))$, we have the following lemma

Lemma 0.2. The map

$$P(V_{univ}^n) \to BU(n)$$

is equivalent to the map

$$B(U(1) \times U(n-1)) \to BU(n).$$

The universal instance of the splitting principle is therefore given by the map induced from the inclusion of the maximal torus $T^n = U(1)^n \le U(n)$:

$$L_{univ} \boxplus \cdots \boxplus L_{univ} \longrightarrow V_{univ}^n$$
$$B(U(1) \times \cdots \times U(1)) \longrightarrow BU(n)$$

We have shown the map

$$\mathbb{Z}[c_1, \ldots, c_n] = H^*(BU(n)) \xrightarrow{\phi} H^*(BU(1)^n) = \mathbb{Z}[x_1, \ldots, x_n]$$

is an injection.

Let NT^n be the normalizer of T^n in $U(n)$. The quotient of NT^n by T^n is called the Weyl group, and is given by

$$NT^n/T^n \cong \Sigma_n.$$

Here, the induced conjugation action of Σ_n on T^n is given by permutation of the factors.

In the homework, you showed that the action of conjugation of an element of G on $H^*(BG)$ was trivial. We deduce that the image of the map ϕ is invariant under the action of Σ_n, giving an injection

$$\mathbb{Z}[c_1, \ldots, c_n] = H^*(BU(n)) \xrightarrow{\phi} H^*(BU(1)^n)^{\Sigma_n} = \mathbb{Z}[x_1, \ldots, x_n]^{\Sigma_n}.$$

The right-hand side is the ring of symmetric polynomials, which is known to be a polynomial algebra on the elementary symmetric polynomials:

$$\mathbb{Z}[x_1, \ldots, x_n]^{\Sigma_n} = \mathbb{Z}[e_1, e_2, \ldots, e_n].$$

Here, e_i is the *ith elementary symmetric polynomial*:

$$e_i(x_1, \ldots, x_n) = \sum_{1 \leq j_1 < \cdots < j_i \leq n} x_{j_1} \cdots x_{j_i}.$$

Explicitly:

$$e_1 = x_1 + x_2 + \cdots + x_n,$$
$$e_2 = x_1 x_2 + x_1 x_3 + \cdots,$$
$$\vdots$$
$$e_n = x_1 x_2 \cdots x_n.$$

Claim 0.3. The inclusion

$$\phi : \mathbb{Z}[c_1, \ldots, c_n] \hookrightarrow \mathbb{Z}[e_1, \ldots, e_n]$$

is an isomorphism. The generator c_i is mapped to e_i.

In fact, Claim 0.3 is equivalent to the sum formula. Indeed, assume we know the sum formula. Let

$$p_i : BU(1)^n \to BU(1)$$

be the projection onto the ith component, and let $L_i = p_i^* L_{univ}$. Then we have

$$\phi(c_i) = c_i(L_{univ} \boxplus \cdots \boxplus L_{univ})$$
$$= c_i(L_1 \oplus \cdots \oplus L_n)$$
$$= \sum_{i_1 + \cdots + i_n = i} c_{i_1}(L_1) \cdots c_{i_n}(L_n)$$
$$= \sum_{1 \leq j_1 < \cdots < j_i \leq n} x_{j_1} \cdots x_{j_i}$$
$$= e_i.$$

Conversely, we can deduce the sum formula from Claim 0.3. Indeed, the splitting principle tells us that we need only prove the sum formula for sums of line bundles. Claim 0.3 does this for the universal example of a sum of line bundles. The general case is then obtained by pullback from the universal example.