18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Epilogue (Lecture 38)

Let $\mathcal{U}$ denote the category of unstable modules over the Steenrod algebra $\mathcal{A}$. In the last two lectures, we defined two different filtrations of $\mathcal{U}$ by Serre classes. On the one hand, we have the classes

$$\ldots \subseteq \mathrm{Nil}_2 \subseteq \mathrm{Nil}_1 \subseteq \mathrm{Nil}_0 = \mathcal{U}$$

such that Nil_i is the kernel of the localization functor

$$\mathcal{U} \xrightarrow{f_n} \mathrm{Fun}_n^{\mathrm{an}}.$$

In other words, Nil_i consists of the collection of all unstable Steenrod modules M such that $(T_V M)^j = 0$ for all $j < i$ and all finite dimensional vector spaces V. (We can also describe Nil_i as the smallest Serre class which contains all i-fold suspensions, though have not proven this.)

On the other hand, we have the Krull filtration

$$\mathrm{Krull}^0 \subseteq \mathrm{Krull}^1 \subseteq \ldots,$$

where Krull^i consists of all unstable $\mathcal{A}$-modules M such that $\overline{T}^i M = 0$.

Our first goal in this lecture is to answer the following three questions:

(A) In what sense do these filtrations "converge" to $\mathcal{U}$?

(B) How do these filtrations interact?

(C) What do the successive quotients look like?

We begin by discussing (A).

Lemma 1. *The canonical functor* $f_\infty : \mathcal{U} \to \varprojlim_n \mathrm{Fun}_n^{\mathrm{an}}$ *is fully faithful.*

Proof. The (2)-limit of the categories Fun_n can be identified with a category of enriched functors $\mathrm{Vect}^f \to \mathcal{U}$. Then f_∞ is defined by the formula

$$(f_\infty M)(V) = T_V M.$$

This functor has a left inverse, given by evaluation at the zero vector space. □

Lemma 2. *The Krull filtration*

$$\mathrm{Krull}^0 \subseteq \mathrm{Krull}^1 \subseteq \ldots,$$

on $\mathcal{U}$ *is exhaustive. In other words, the smallest Serre class containing each* Krull^i *is* $\mathcal{U}$ *itself.*

Proof. The category $\mathcal{U}$ is generated under colimits by the free unstable modules $F(n)$. We will show that $F(n) \in \mathrm{Krull}^n$ for each $n \geq 0$, using induction on n. Recall that we computed

$$TF(n) \simeq F(n) \oplus F(n-1) \oplus \ldots \oplus F(0).$$

Consequently, $\overline{T}F(n) \simeq F(n-1) \oplus \ldots \oplus F(0)$. By the inductive hypothesis, $\overline{T}F(n) \in \mathrm{Krull}^{n-1}$, so $\overline{T}^{n+1}F(n) \simeq \overline{T}^n \overline{T}F(n) \simeq 0$ as desired. □

We now address question (B). For each $n \geq 0$, we can define a "shift" functor S from $\mathrm{Fun}_n^{\mathrm{an}}$ to itself by the formula $S(G)(V) = G(V \oplus \mathbf{F}_2)$. By construction, the diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "T"] \arrow[d, "f_n"] & \mathcal{U} \arrow[d, "f_n"] \\ \mathrm{Fun}_n^{\mathrm{an}} \arrow[r, "S"] & \mathrm{Fun}_n^{\mathrm{an}} \end{tikzcd}$$

commutes up to isomorphism. We note that the functor $S(G)$ contains G as a retract (since $V \oplus \mathbf{F}_2$ contains V as a retract); we therefore have $S(G) = G \oplus \Delta G$, where Δ is another functor from $\mathrm{Fun}_n^{\mathrm{an}}$ to itself fitting into a commutative diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\overline{T}"] \arrow[d, "f_n"] & \mathcal{U} \arrow[d, "f_n"] \\ \mathrm{Fun}_n^{\mathrm{an}} \arrow[r, "\Delta"] & \mathrm{Fun}_n^{\mathrm{an}} \end{tikzcd}$$

By definition, the kernel of Δ^{k+1} can be identified with the subcategory $\mathrm{Fun}_n^{(k)} \subseteq \mathrm{Fun}_n^{\mathrm{an}}$ consisting of functors which are polynomial of degree $\leq k$. One can show that $\mathrm{Fun}_n^{(k)}$ is precisely the image of Krull^k in Fun_n, so that the Krull filtration on $\mathcal{U}$ induces the filtration

$$\mathrm{Fun}_n^{(0)} \subseteq \mathrm{Fun}_n^{(1)} \subseteq \ldots$$

on $\mathrm{Fun}_n^{\mathrm{an}}$.

Warning 3. This is *not* the Krull filtration on the category Fun_n. In fact, we have seen that every Noetherian object of Fun_n has finite length, so that $\mathrm{Krull}^0(\mathrm{Fun}_n) = \mathrm{Fun}_n$.

We now address question (C). We begin by considering the associated graded of the Nil-filtration:

Proposition 4. *The iterated suspension functor Σ^n induces an equivalence of categories*

$$\mathcal{U} \,/\, \mathrm{Nil}_1 \to \mathrm{Nil}_n \,/\, \mathrm{Nil}_{n+1}$$

Proof. We can identify $\mathcal{U} \,/\, \mathrm{Nil}_{n+1}$ with the category $\mathrm{Fun}_n^{\mathrm{an}}$ of enriched analytic functors from Vect^f to $\mathcal{U}^{\leq n}$. The Serre class $\mathrm{Nil}_n \,/\, \mathrm{Nil}_{n+1}$ can be identified with the kernel of the further localization obtained by composing these functors with the truncation $\tau^{\leq n-1} : \mathcal{U}^{\leq n} \to \mathcal{U}^{\leq n-1}$. This is equivalent to functors which land in the category $\mathcal{U}^{=n}$ of unstable $\mathcal{A}$-modules which are concentrated in degree n. But this category can be identified with the category of vector spaces, via the functor $V \mapsto \Sigma^n V$. $\square$

We will therefore restrict our attention to the category $\mathrm{Fun}^{\mathrm{an}} = \mathrm{Fun}_0^{\mathrm{an}}$ consisting of analytic functors from Vect^f to Vect. This has a filtration by subcategories

$$\mathrm{Fun}^{(0)} \subseteq \mathrm{Fun}^{(1)} \subseteq \ldots$$

where $\mathrm{Fun}^{(n)}$ denotes the class of polynomial functors of degree $\leq n$.

Example 5. Let R be a represenation of the symmetric group Σ_n. Then the functor

$$V \mapsto (V^{\otimes n} \otimes R)_{\Sigma_n}$$

is a polynomial functor of degree n, which we will denote by F_R.

The structure of the homogeneous layers $\mathrm{Fun}^{(n)} \,/\, \mathrm{Fun}^{(n-1)}$ can be described by the following result:

Proposition 6. *Let* Mod_{Σ_n} *denote the category of modules over the group ring* $\mathbf{F}_2[\Sigma_n]$*. Then the construction*

$$R \mapsto F_R$$

defines a functor $\mathrm{Mod}_{\Sigma_n} \to \mathrm{Fun}^{(n)}$*. Moreover, the composition*

$$\mathrm{Mod}_{\Sigma_n} \to \mathrm{Fun}^{(n)} \to \mathrm{Fun}^{(n)} / \mathrm{Fun}^{(n-1)}$$

is an equivalence of categories.

Proof. Let F be a polynomial functor of degree $\leq n$. Let S be a set of cardinality n, and let $\mathbf{F}_2^S$ denote the corresponding n-dimensional vector space over $\mathbf{F}_2$. Let R be the kernel of the map

$$F(\mathbf{F}_2^S) \to \prod_{s \in S} F(\mathbf{F}_2^{S - \{s\}}).$$

Then R is a vector space over $\mathbf{F}_2$, equipped with an action of the symmetric group Σ_n of permutations of S. Moreover, if F has degree $< n$, then R vanishes. This construction furnishes a functor $\mathrm{Fun}^{(n)} / \mathrm{Fun}^{(n-1)} \to \mathrm{Mod}_{\Sigma_n}$ which is inverse to the construction above. □

We can summarize our results as follows:

Theorem 7. (1) *The category* $\mathcal{U}$ *admits a filtration by Serre classes*

$$\ldots \subseteq \mathrm{Nil}_2 \subseteq \mathrm{Nil}_1 \subseteq \mathrm{Nil}_0 = \mathcal{U}.$$

Moreover, $\mathcal{U}$ *embeds fully faithfully into the inverse limit* $\varprojlim \mathcal{U} / \mathrm{Nil}_n$*, and the successive quotients* $\mathrm{Nil}_n / \mathrm{Nil}_{n+1}$ *are equivalent to* $\mathrm{Fun}^{\mathrm{an}}$.

(2) *The Krull filtration on* $\mathcal{U}$ *induces a filtration on each* $\mathrm{Nil}_n / \mathrm{Nil}_{n+1}$*, which can be identified with the filtration of* $\mathrm{Fun}^{\mathrm{an}}$ *by polynomial functors*

$$\mathrm{Fun}^{(0)} \subseteq \mathrm{Fun}^{(1)} \subseteq \ldots$$

(3) *Each successive quotient* $\mathrm{Fun}^{(n)} / \mathrm{Fun}^{(n-1)}$ *can be identified with the category of representations of the symmetric group* Σ_n *(in the category of* $\mathbf{F}_2$*-vector spaces).*

We conclude this lecture (and this course) with a digression on another topic. We have shown that if G is a finite p-group, then the classifying space BG is an atomic object in the category of p-profinite spaces $\mathfrak{S}_p^\vee$. However, we have also shown that BG is *not* atomic in the category of spaces unless G is the trivial group. Nevertheless, some consequences of the atomicity of G still carry over to the setting of spaces: for example, we used the atomicity of BG in $\mathfrak{S}_p^\vee$ to show that every map from BG into a simply connected finite complex is nullhomotopic.

One might try to prove that BG is atomic in $\mathfrak{S}$ using the same techniques. Of course, such an attempt is doomed to failure, but might still teach us something or yield a weaker result. The basic idea is simple: given an arbitrary space X, we can attempt to compute the mapping space X^{BG} using an arithmetic square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p \widehat{X}_p \arrow[d] \\ X_{\mathbf{Q}} \arrow[r] & (\prod_p \widehat{X}_p)_{\mathbf{Q}}, \end{tikzcd}$$

and show that each term in this square behaves well with respect to pushouts in X. Of course, there are potentially many difficulties:

(1) The space X might fail to be simply connected.

(2) The space X might fail to be of finite type, in which case the p-profinite completion is not the appropriate thing to put into the arithmetic square.

(3) We might not be able to assemble local information about pushout squares into global information, since homotopy pushouts and homotopy pullbacks generally do not commute.

Rather than address these questions, we want to discuss a classical result which suggests that problem (1) is not as difficult as it seems:

Theorem 8. *Let G and H be groups and $G \star H$ their free product. Let F be any finite group. Then any homomorphism $\phi : F \to G \star H$ is either conjugate to a homomorphism from F into G or conjugate to a homomorphism from F into H.*

Remark 9. We can rephrase this result in terms of homotopy theory: any map

$$BF \to BG \vee BH$$

is homotopic to a map from BF into BG or to a map from BF into BH. This is a kind of "atomicity" property enjoyed by the classifying space BF.

Proof. We define a bipartite graph X as follows: the vertex set of X is $V_0 \coprod V_1$, where $V_0 = (G \star H)/G$ and $V_1 = (G \star H)/H$. The edge set of X is $G \star H$, where an element $g \in G \star H$ determines an edge from $gG \in V_0$ to $vH \in V_1$. The graph X is a tree, which admits an action of $G \star H$ by left translation. Given a homomorphism $\phi : F \to G \star H$, we see that the finite group F acts on the tree X. By the Bruhat-Tits fixed point theorem, the fixed point set X^F is nonempty. In particular, X^F contains a vertex of X. Without loss of generality, we may assume that this vertex has the form gG, for $g \in G \star H$. In this case, conjugation by g carries ϕ to a homomorphism $F \to G$. $\square$

Of course, this is only the tip of the iceberg as far as what can be proven using these sorts of techniques. For example, a more elaborate version of the same proof can be used to show the following:

- Suppose given injections of groups

$$G_0 \hookleftarrow G \hookrightarrow G_1.$$

Let F be a finite group. Then the diagram

$$\begin{tikzcd} (BG)^{BF} \arrow[r] \arrow[d] & (BG_0)^{BF} \arrow[d] \\ (BG_1)^{BF} \arrow[r] & B(G_0 \coprod_G G_1)^{BF} \end{tikzcd}$$

is a homotopic pushout square.

This raises the following question: exactly how close is BF to being atomic in the category of spaces? It seems likely that a satisfying answer to this question will involve *both* the sort of combinatorial group theory argument sketched above, and the technology of unstable Steenrod modules developed in these lectures.