18.950 Differential Geometry
Fall 2008

CHAPTER 3

Global geometry of hypersurfaces

Lecture 24

Definition 24.1. A *hypersurface* is a subset $M \subset \mathbb{R}^{n+1}$ with the following property. For every $y \in M$ there is an open subset $V \subset \mathbb{R}^{n+1}$ containing y, and a function $\psi : V \to \mathbb{R}$ whose zero set $\psi^{-1}(0)$ is precisely $M \cap V$, and whose derivative is nonzero at any point of $M \cap V$.

Example 24.2. $M = \{x_1 x_2 x_3 = 0\} \subset \mathbb{R}^{n+1}$ is not a hypersurface.

We call ψ a *local defining function* for M near y. We are looking for properties of M that are independent of how it is presented as a zero set. The following is useful:

Proposition 24.3 (L'Hopital's rule; proof postponed). Let $\psi : V \to \mathbb{R}$ be a local defining function for M, and $\phi : V \to \mathbb{R}$ another smooth function which vanishes along $V \cap M$. Then there a unique smooth function $q : V \to \mathbb{R}$ such that $\phi = q\psi$.

Corollary 24.4. Let $\psi, \tilde{\psi} : V \to \mathbb{R}$ be two local defining functions for M. Then there is a unique smooth nowhere vanishing function $q : V \to \mathbb{R}$ such that $\tilde{\psi} = q\psi$.

Let M be a hypersurface, $\psi : V \to \mathbb{R}$ a local defining function, and $y \in V \cap M$ any point. The derivative $D\psi_y$ is independent of the choice of ψ up to multiplication with a nonzero real number. Hence, its nullspace

$$TM_y = \ker D\psi_y = \{Y \in \mathbb{R}^{n+1} \mid D\psi_y \cdot Y = \langle \nabla\psi_y, Y \rangle = 0\}$$

is independent of ψ. We call it the *tangent space* of M at y.

Example 24.5. The unit sphere $S^n = \{\|y|^2 = 1\} \subset \mathbb{R}^{n+1}$ is a hypersurface, and $TS^n_y = y^\perp$.

Lecture 25

Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface. If ψ is a local defining function for M, then the map $\nabla\psi/\|\nabla\psi\| : M \cap V \to S^n$ is independent of the choice of ψ up to sign. This ambiguity will be further discussed later. Similarly, for all $y \in M \cap V$, the linear map

$$L_y : TM_y \longrightarrow TM_y, \;\; L_y(Y) = -D(\nabla\psi/\|\nabla\psi\|)_y \cdot Y$$

is independent of the choice of ψ up to sign. We call it the *shape operator* of M at y.

Example 25.1. Take the hyperboloid $M = \{y_1^2 = y_2^2 + y_3^2 + 1\}$ in $\mathbb{R}^3$ (this is Euclidean space, and not to be confused with curvature computations in Minkowski space). The tangent space at any point y is spanned by $Y_1 = (y_2, y_1, 0)$ and $Y_2 = (y_3, 0, y_1)$. The matrix of $D(\nabla\psi/\|\nabla\psi\|)_y : TM_y \to TM_y$ with respect to this basis is

$$\frac{1}{\|y\|^3}\begin{pmatrix} y_1^2 - y_2^2 + y_3^2 & -2y_2y_3 \\ -2y_2y_3 & y_1^2 - y_3^2 + y_2^2 \end{pmatrix}$$

Lemma 25.2. For $X, Y \in TM_y$,

$$\langle X, L_y \cdot Y\rangle = -\frac{1}{\|\nabla_y\psi\|}\langle X, D^2\psi_y \cdot Y\rangle.$$

This proves that L_y is selfadjoint with respect to the standard inner product $\langle\cdot,\cdot\rangle$ on TM_y. In particular, it has real eigenvalues, which we call the principal curvatures. The mean, Gauss, and scalar curvature are then defined as usual. Finally, the Riemann curvature operator is defined to be

$$\mathcal{R}_y = \Lambda^2(L_y) : \Lambda^2(TM_y) \longrightarrow \Lambda^2(TM_y).$$

Note that $\Lambda^2(L_y) = \Lambda^2(-L_y)$, so the Riemann curvature operator does not suffer from sign ambiguities. The same applies to the Gauss curvature if n is even.

Example 25.3. The Gauss curvature of the hyperboloid discussed above is the determinant of L_y, which is $(2y_1^2 - 1)/\|y\|^6$.

Proposition 25.4. Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface. Take a point $y \in M$, a local defining function ψ for M near y, and an orthonormal basis $Y_1, \dots, Y_n$ of the tangent space TM_y.

$$\kappa_{mean} = \pm\frac{1}{\|\nabla\psi_y\|}\sum_{i=1}^{n}\langle Y_i, D^2\psi_y \cdot Y_i\rangle,$$

$$\kappa_{gauss} = \pm\frac{\det(D^2\psi_y \cdot Y_1, \dots, D^2\psi_y \cdot Y_n, \nabla\psi_y)}{\|\nabla\psi_y\|^{n+1}}.$$

Lecture 26

Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface.

DEFINITION 26.1. A function $f : M \to \mathbb{R}$ if smooth if for every point $y \in M$ there is an open subset $V \subset \mathbb{R}^{n+1}$ containing y, and a smooth function $\tilde{f} : V \to \mathbb{R}$, such that $f|M \cap V = \tilde{f}|M \cap V$. We call $\tilde{f}$ a *local extension* of f.

The derivative Df_y is the linear map $TM_y \to \mathbb{R}$ defined by $Df_y = D\tilde{f}_y \,|\, TM_y$. This is independent of the choice of local extension.

This generalizes to several functions, which means to maps $f : M \to \mathbb{R}^{k+1}$. If $M \subset \mathbb{R}^{n+1}$ and $N \subset \mathbb{R}^{k+1}$ are hypersurfaces, and $f : M \to \mathbb{R}^{k+1}$ a smooth map whose image lies in N, then the image of the derivative Df_y lies in the tangent space to N at $f(y)$. Hence, Df_y is a linear map $TM_y \to TN_{f(y)}$.

DEFINITION 26.2. A *Gauss map* for M is a smooth map $\nu : M \to \mathbb{R}^{n+1}$ such that for all y, $\nu(y)$ is of unit length and orthogonal to TM_y.

We also call a Gauss map a choice of orientation. Suppose that such a ν is given. If ψ is a local defining function for M, we have $\nu = \pm\nabla\psi/\|\nabla\psi\|$ on $M \cap V$, where the sign is locally constant. If the sign is positive everywhere, we say that ψ is compatible with the choice of orientation.

Consider the derivative of the Gauss map, which is

$$D\nu_y : TM_y \longrightarrow T(S^n)_{\nu(y)} = \nu(y)^\perp = TM_y.$$

We re-define the shape operator to be $L_y = -D\nu_y$. This agrees with the previous definition, except that the sign ambiguity has been removed by the choice of orientation.

REMARK 26.3. In Proposition 25.4, assume that M is oriented, and ψ compatible with the choice of orientation. Then the sign in κ_{mean} is -1. Assume in addition that the basis is chosen in such a way that $\det(Y_1, \dots, Y_n, \nabla\psi_y) > 0$. Then the sign in κ_{gauss} is $(-1)^n$.

Lecture 27

DEFINITION 27.1. Let U, V be open subsets of $\mathbb{R}^{n+1}$. A smooth map $\phi : V \to U$ is a *diffeomorphism* if it is one-to-one and the inverse ϕ^{-1} is also smooth (it is in fact enough to check that ϕ is one-to-one and that $D\phi_y$ is invertible for all y, since that ensures smoothness of the inverse).

One can think of diffeomorphisms as curvilinear coordinate changes.

THEOREM 27.2 (Inverse function theorem; no proof). Let $\tilde{V} \subset \mathbb{R}^{n+1}$ be an open subset, $y \in \tilde{V}$ a point, and $\phi : \tilde{V} \to \mathbb{R}^{n+1}$ a smooth map such that $D\phi_y$ is invertible. Then there is an open subset $V \subset \tilde{V}$, still containing y, such that: $U = \phi(V)$ is open, and $\phi|V : V \to U$ is a diffeomorphism.

COROLLARY 27.3 (Implicit function theorem, special case). Let $\tilde{V} \subset \mathbb{R}^{n+1}$ be an open subset, $\psi : \tilde{V} \to \mathbb{R}$ a smooth function, and $y \in V$ a point such that $\psi(y) = 0$, $D\psi(y) \neq 0$. Then there are: an open subset $V \subset \tilde{V}$, still containing y; an open subset $U \subset \mathbb{R}^{n+1}$ containing 0; and a diffeomorphism $\phi : U \to V$ such that $\phi(0) = y$, and $\psi(\phi(x)) = x_{n+1}$ for all x.

The informal meaning is that in the curvilinear local coordinate system ϕ, ψ looks like a linear function.

Lecture 28

Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface. Take a local defining function $\psi : V \to \mathbb{R}$, defined near some point $y \in V$. The derivative $D\psi_y : \mathbb{R}^{n+1} \to \mathbb{R}$ depends on ψ, but its nullspace

$$\ker(D\psi_y) = \{X \in \mathbb{R}^{n+1} \mid D\psi_y \cdot X = \langle \nabla\psi_y, X\rangle = 0\}$$

does not, since it is the tangent space TM_y. Now assume that M comes with a Gauss vector, which on $V \cap M$ agrees with $\nabla\psi/\|\nabla\psi\|$. The Hessian $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$, $(X, Y) \mapsto \langle X, D^2\psi_y Y\rangle$ depends on Y, but the map

$$TM_y \times TM_y \longrightarrow \mathbb{R}, \;\; (X, Y) \longmapsto \frac{1}{\|\nabla\psi\|}\langle X, D^2\psi_y Y\rangle$$

is independent of ψ, since it can be written as $-\langle X, LY\rangle$ where $L : TM_y \to TM_y$ is the shape operator.

Example 28.1. Take the hyperboloid $M = \{y_1^2 = y_2^2 + y_3^2 + 1\}$ in $\mathbb{R}^3$ (this is Euclidean space, and not to be confused with curvature computations in Minkowski space). A Gauss normal is

$$\nu(y) = \frac{1}{\|y\|}(-y_1, y_2, y_3).$$

The tangent space at any point y is spanned by $Y_1 = (y_2, y_1, 0)$ and $Y_2 = (y_3, 0, y_1)$. The matrix of $D\nu_y : TM_y \to TM_y$ with respect to this basis is

$$\frac{1}{\|y\|^3}\begin{pmatrix} y_1^2 - y_2^2 + y_3^2 & -2y_2y_3 \\ -2y_2y_3 & y_1^2 - y_3^2 + y_2^2 \end{pmatrix}$$

The Gauss curvature is its determinant, which is $(2y_1^2 - 1)/\|y\|^6$. In particular, it's always positive.

Let U, V be open subsets of $\mathbb{R}^{n+1}$. A smooth map $\phi : V \to U$ is a *diffeomorphism* if it is one-to-one and the inverse ϕ^{-1} is also smooth (it is in fact enough to check that ϕ is one-to-one and that $D\phi_y$ is invertible for all y, since that ensures smoothness of the inverse). One can think of diffeomorphisms as curvilinear coordinate changes.

Theorem 28.2 (Inverse function theorem; no proof). Let $\tilde{V} \subset \mathbb{R}^{n+1}$ be an open subset, $y \in \tilde{V}$ a point, and $\phi : \tilde{V} \to \mathbb{R}^{n+1}$ a smooth map such that $D\phi_y$ is invertible. Then there is an open subset $V \subset \tilde{V}$, still containing y, such that: $U = \phi(V)$ is open, and $\phi|V : V \to U$ is a diffeomorphism.

Lecture 29

Corollary 29.1 (Implicit function theorem, special case). Let $\tilde{V} \subset \mathbb{R}^{n+1}$ be an open subset, $\psi : \tilde{V} \to \mathbb{R}$ a smooth function, and $y \in \tilde{V}$ a point such that $\psi(y) = 0$, $D\psi(y) \neq 0$. Then there are: an open subset $V \subset \tilde{V}$, still containing y; an open subset $U \subset \mathbb{R}^{n+1}$ containing 0; and a diffeomorphism $\phi : U \to V$ such that $\phi(0) = y$, and $\psi(\phi(x)) = x_{n+1}$ for all x.

The informal meaning is that in the curvilinear local coordinate system ϕ, ψ looks like a linear function.

Lemma 29.2. Let $U \subset \mathbb{R}^{n+1}$ be an open subset contaning the origin, and $\psi : U \to \mathbb{R}$ a smooth function which vanishes at all points $x \in U$ whose last coordinate x_{n+1} is zero. Then there is a unique smooth function q such that $\psi = qx_{n+1}$.

This and the previous Corollary together imply our version of l'Hopital's theorem (Lemma 20.2).

Definition 29.3. Let M be a hypersurface. A *partial parametrization* of M consists of an open subset $V \subset \mathbb{R}^{n+1}$, an open subset $U \subset \mathbb{R}^n$, and a hypersurface patch $f : U \to \mathbb{R}^{n+1}$ which is one-to-one (injective), and whose image is precisely $M \cap V$.

Corollary 29.4. For every point $y \in M$, there is a partial parametrization such that $y \in f(U) = M \cap V$.

If f is a partial parametrization, the $\partial_{x_i} f$ form a basis of $TM_{f(x)}$ for all x. Equivalently, $Df_x : \mathbb{R}^n \to TM_{f(x)}$ is an isomorphism (an invertible linear map). In the case where M comes with a Gauss vector field $\nu : M \to \mathbb{R}^{n+1}$, one can always choose these partial parametrizations to be compatible with it, which means that $\det(\partial_{x_1} f, \dots, \partial_{x_n} f, \nu(f(x))) > 0$.

Proposition 29.5. Let f be a partial parametrization. Denote by I^f its first fundamental form, and by S^f its shape operator. Under the identification $Df : \mathbb{R}^n \to TM_{f(x)}$, I^f turns into the ordinary scalar product, and S^f into the shape operator S of M.

Explicitly, the second part of this says that $S : TM_{f(x)} \to TM_{f(x)}$ and $S^f : \mathbb{R}^n \to \mathbb{R}^n$ are related by

$$S^f = Df^{-1} \cdot S \circ Df.$$

Lecture 30

Proposition 30.1. Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface. Take a point $y \in M$, a local defining function ψ for M near y, and an orthonormal basis $Y_1, \dots, Y_n$ of the tangent space TM_y.

$$\kappa_{mean} = \pm \frac{1}{\|\nabla \psi_y\|} \sum_{i=1}^{n} \langle Y_i, D^2\psi_y \cdot Y_i \rangle,$$
$$\kappa_{gauss} = \pm \frac{\det(D^2\psi_y \cdot Y_1, \dots, D^2\psi_y \cdot Y_n, \nabla \psi_y)}{\|\nabla \psi_y\|^{n+1}}.$$

Assume that $\nu(y) = \nabla\psi_y / \|\nabla\psi_y\|$. Then the sign in κ_{mean} is -1. Assume in addition that the basis is chosen in such a way that $\det(Y_1, \dots, Y_n, \nabla\psi_y) > 0$. Then the sign in κ_{gauss} is $(-1)^n$.

Definition 30.2. A hypersurface $M \subset \mathbb{R}^{n+1}$ is compact if it is bounded and closed (closedness means that if a sequence $y_n \in M$ converges to some point $y_\infty \in \mathbb{R}^{n+1}$, then that point must also lie in M).

Definition 30.3. A hypersurface $M \subset \mathbb{R}^{n+1}$ is connected if every smooth function $\phi : M \to \mathbb{R}$ whose derivative is identically zero is actually constant.

Theorem 30.4 (from topology; no proof). A connected compact hypersurface is always orientable (in fact, there are precisely two choices of Gauss vectors, differing by a sign).

Take a connected compact hypersurface, oriented inwards. Then there is a point where all principal curvatures are > 0. Similarly, for the outwards orientation, there is a point where all principal curvatures are < 0. This follows from Example 14.3.

Theorem 30.5 (from topology; no proof). Let $M \subset \mathbb{R}^{n+1}$ be a connected compact hypersurface, with $n \geq 2$, and $\phi : M \to S^n$ a smooth map such that $D\phi_y : TM_y \to TS^n_{\phi(y)}$ is an isomorphism for all y. Then ϕ is bijective (one-to-one and onto).

Definition 30.6. A hypersurface M is convex if for all $y \in M$, the whole of M lies on one side of the hyperplane $y + TM_y$.

We already know from Example 14.2 that if M is compact connected and convex, its principal curvatures any any point are either ≥ 0 (for the orientation pointing inwards) or ≤ 0 (for the orientation pointing outwards).

Theorem 30.7 (Hadamard). Let $M \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a compact connected hypersurface, whose Gauss curvature is everywhere nonzero. Then M is convex.

REMARK 30.8. For a compact connected hypersurface $M \subset \mathbb{R}^{n+1}$, $n \geq 2$, the following are equivalent: (i) the Gauss curvature is everywhere nonzero; (ii) the Riemann curvature operator has only positive eigenvalues everywhere; (ii) the principal curvatures are either everywhere > 0 or everywhere < 0.

Lecture 31

Let $M \subset \mathbb{R}^{n+1}$ be a compact hypersurface, and $\phi : M \to \mathbb{R}$ a smooth function. We want to quickly sketch the definition of the integral of ϕ. Recall that the *support* $\mathrm{supp}(\phi) \subset M$ is the closure of the set of points where ϕ is nonzero. First suppose that ϕ has *small support*, which means that *supp*(ϕ) is contained in the image of a partial parametrization $f : U \to M$, and write $\phi^f = \phi \circ f : U \to \mathbb{R}$. In that case,

$$\int_M \phi(y)\, d\mathrm{vol}_y \stackrel{\mathrm{def}}{=} \int_U \phi^f \sqrt{\det(G^f)}\, dx,$$

This makes sense because it's invariant under diffeomorphisms. For general ϕ, there are two equivalent ways: either write it as $\phi = \phi_1 + \cdots + \phi_m$ where each ϕ_i has small support. Then,

$$\int_M \phi(y)\, d\mathrm{vol}_y \stackrel{\mathrm{def}}{=} \sum_{i=1}^{m} \int_M \phi_i(y)\, d\mathrm{vol}_y.$$

Alternatively, suppose that M is decomposed into polytopes in the following sense. There is a collection of partial parametrizations $f_i : U_i \to M$ and polytopes $P_i \subset U_i$ $(1 \leq i \leq m)$, such that $M = f_1(P_1) \cup \cdots \cup f_m(P_m)$, and with the interiors $f_i(P_i \setminus \partial P_i)$ pairwise disjoint. Then

$$\int_M \phi(y)\, d\mathrm{vol}_y \stackrel{\mathrm{def}}{=} \sum_{i=1}^{m} \int_{P_i} \phi^{f_i} \sqrt{\det(G^{f_i})}\, dx,$$

where ϕ^{f_i} and G^{f_i} are defined as before.

Lemma 31.1. Let f be a partial parametrization, and ν^f the associated Gauss normal. Then $\det(G^f) = \det(\partial_{x_1} f, \dots, \partial_{x_n} f, \nu^f)^2$. In particular, in the case of a surface,

$$\sqrt{\det(G^f)} = \|\partial_{x_1} f \times \partial_{x_2} f\|.$$

Example 31.2. The volume of M is defined as $\mathrm{vol}(M) = \int_M 1 d\mathrm{vol}$.

Let $M, \tilde{M}$ be hypersurfaces in $\mathbb{R}^{n+1}$, and $\phi : M \to \tilde{M}$ a smooth map. Suppose that both our hypersurfaces come with Gauss normal vectors $\nu, \tilde{\nu}$. We then define $\det(D\phi_y)$ by writing $D\phi_y : TM_y \to T\tilde{M}_{\phi(y)}$ in terms of orthonormal bases of those vector spaces which are compatible with the orientation. This means:

Definition 31.3. In the situation above, let $(X_1, \dots, X_n)$ be a basis of TM_y such that $\det(X_1, \dots, X_n, \nu(y)) > 0$, and $(Y_1, \dots, Y_n)$ a basis of $T\tilde{M}_{\phi(y)}$ such that $\det(Y_1, \dots, Y_n, \tilde{\nu}(\phi(y))) > 0$. Take the matrix A such that $D\phi_y(X_i) = \sum_j A_{ji} Y_j$, and define $\det(D\phi_y) = \det(A)$. This is independent of the choices of bases.

EXAMPLE 31.4. Consider the Gauss map $\nu : M \to \tilde{M} = S^n$, where S^n carries a Gauss normal vector $\nu(y) = y$. Then $\det(D\nu_y)$ is $(-1)^n$ times the Gauss curvature of M at y.

Lecture 32

LEMMA 32.1. Let $M, \tilde{M}$ be hypersurfaces, with Gauss maps $\nu, \tilde{\nu}$, and $\phi : M \to \tilde{M}$ be a smooth map. Suppose that we have a parametrization $f : U \to M$ compatible with the orientation. Set $\phi^f = \phi \circ f : U \to \tilde{M} \subset \mathbb{R}^{n+1}$, and let G^f be the first fundamental form. Then for $y = f(x)$,

$$\det(D\phi)_y = \frac{\det(\partial_{x_1}\phi^f, \dots, \partial_{x_n}\phi^f, \tilde{\nu}(\phi^f(x)))}{\sqrt{\det(G^f(x))}}.$$

DEFINITION 32.2. Let $M, \tilde{M}$ be compact hypersurfaces equipped with Gauss maps. Assume that $\tilde{M}$ is connected. Let $\phi : M \to \tilde{M}$ be a smooth map. The *degree* of ϕ is defined as

$$\deg(\phi) = \frac{1}{\operatorname{vol}(\tilde{M})} \int_M \det(D\phi_y)\, d\mathrm{vol}_y.$$

PROPOSITION 32.3. Suppose that $\tilde{M}$ is decomposed into $f_i(P_i)$ as in the previous lecture, where $f_i : U_i \to M$ are partial parametrization, and $P_i \subset U_i$ polytopes. Then

$$\deg(\phi) = \frac{1}{\operatorname{vol}(\tilde{M})} \left(\sum_i \int_{P_i} \det(\partial_{x_1}\phi^{f_i}, \dots, \partial_{x_n}\phi^{f_i}, \tilde{\nu}(\phi(f_i(x))))\, dx \right).$$

where $\phi^{f_i} = \phi \circ f_i$.

LEMMA 32.4 (Sketch proof). Suppose that ϕ is bijective (one-to-one and onto), and that $\det(D\phi)$ is everywhere positive (or everywhere negative). Then $\deg(\phi) = 1$ (or -1).

THEOREM 32.5 (No proof). The degree is always an integer.

Lecture 33

EXAMPLE 33.1. Let $M \subset \mathbb{R}^3$ be a torus, parametrized by

$$f(x_1, x_2) = ((\cos x_1)(2 + \cos x_2), (\sin x_1)(2 + \cos x_2), \sin x_2)$$

In this parametrization, the first fundamental form is

$$G = \begin{pmatrix} (2+\cos x_2)^2 & 0 \\ 0 & 1 \end{pmatrix},$$

hence $\sqrt{\det G} = 2 + \cos x_2$ and

$$\mathrm{vol}(M) = 8\pi^2.$$

Take the map $\phi : M \to M$ which wraps the torus twice around itself, sending $f(x_1, x_2)$ to $f(2x_1, x_2)$. Then $\det(D\phi) = 2$ everywhere, hence $\deg(\phi) = 2$.

Now consider the map $\tilde{\phi} : M \to M$ wrapping the other way, which means that it sends $f(x_1, x_2)$ to $f(x_1, 2x_2)$. With respect to the orthonormal basis $(\partial_{x_1} f/(2 + \cos x_2), \partial_{x_2} f)$, we have

$$D\tilde{\phi}_{f(x_1,x_2)} = \begin{pmatrix} \frac{2+\cos 2x_2}{2+\cos x_2} & 0 \\ 0 & 2 \end{pmatrix},$$

hence $\det(D\tilde{\phi})_{f(x_1,x_2)} = 4\frac{1+\cos x_2}{2+\cos x_2}$, and

$$\int_M \det(D\tilde{\phi})\, d\mathrm{vol} = \int_{[0,2\pi]\times[0,2\pi]} 4(1 + \cos x_2) = 16\pi^2,$$

which means that again $\deg(\tilde{\phi}) = 2$. One can get the same integral formula a little more easily by using Proposition 26.3.

Since the degree is an integer, it is constant under smooth deformations of a map. By applying this idea (called the *homotopy method*), we get:

LEMMA 33.2. Let $M \subset \mathbb{R}^{n+1}$ be a compact hypersurface with a Gauss map, and $\phi : M \to S^n$ a smooth map. If $\deg(\phi) \neq 0$, then ϕ is necessarily onto.

The result generalizes to targets other than S^n, and there is an even more general formula:

THEOREM 33.3 (no proof). Let $M, \tilde{M} \subset \mathbb{R}^{n+1}$ be compact connected hypersurfaces with orientations, and $\phi : M \to \tilde{M}$ a smooth map. Suppose that $p \in \tilde{M}$ is a point with the following properties: (i) there are only finitely many $y_1, \dots, y_k \in M$ such that $\phi(y_i) = p$; (ii) at each y_i, we have $\det(D\phi_{y_i}) \neq 0$. Then

$$\deg(\phi) = \sum_{i=1}^{k} \mathrm{sign}(\det(D\phi_{y_i})).$$

DEFINITION 33.4. Let M be a compact hypersurface with an orientation. The *total Gauss curvature* is

$$\kappa_{gauss}^{tot} = \int_M \kappa_{gauss}\, d\text{vol}.$$

For even-dimensional hypersurfaces, the choice of orientation is actually irrelevant. If we take $\phi = \nu : M \to S^n$ to be the Gauss map, and orient S^n pointing outwards, then $\det(D\phi_y) = \det(-L_y) = (-1)^n \kappa_{gauss}$, hence:

COROLLARY 33.5. Let M be a compact hypersurface with an orientation. Then

$$\kappa_{gauss}^{tot} = (-1)^n \text{vol}(S^n) \deg(\nu).$$

In particular, the total Gauss curvature is always an integer multiple of $\text{vol}(S^n)$.

Lecture 34

We already saw that if $M \subset \mathbb{R}^3$ is a torus, then $\kappa_{gauss}^{tot} = 0$, irrespective of how it's embedded. To generalize this to other surfaces, we need to return to our discussion of moving frames.

DEFINITION 34.1. Let $f : U \to \mathbb{R}^3$ be a surface patch, whose domain contains the origin. Let (X_1, X_2) be a moving frame defined on $U \setminus \{0\}$. We say that the frame has a *singularity of multiplicity* $m \in \mathbb{Z}$ at 0 if it can be written as

$$X_1 = \cos(m\theta)\tilde{X}_1 - \sin(m\theta)\tilde{X}_2,$$
$$X_2 = \sin(m\theta)\tilde{X}_1 + \cos(m\theta)\tilde{X}_2$$

where θ is the angular coordinate, and $(\tilde{X}_1, \tilde{X}_2)$ is a moving frame which extends smoothly over $x = 0$. Passing to the matrices whose column vectors are the X_k and $\tilde{X}_k$, one can write the relation as

$$X = \tilde{X} \exp(m\theta J),$$

where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ as usual.

Let X be a moving frame with a singularity of order m. Last time we considered the vector field

$$\alpha = ((A_1)_{12}, (A_2)_{12}) : U \setminus \{0\} \to \mathbb{R}^2,$$

which was such that $\mathrm{curl}(\alpha) = \kappa_{gauss}\sqrt{\det(G)}$. A computation shows that

$$\alpha = m(x_2, -x_1)/\|x\|^2 + \textit{something bounded in } x,$$

and therefore:

LEMMA 34.2.

$$\lim_{\rho \to 0} \oint_{|x|=\rho} \alpha = -2\pi m.$$

DEFINITION 34.3. Let $M \subset \mathbb{R}^3$ be a compact surface. A *moving frame with singularities* is given by a finite set of points $\{p_1, \dots, p_k\}$ on M, together with maps $Y_1, Y_2 : M \setminus \{p_1, \dots, p_k\} \to \mathbb{R}^3$ which at each point y form a positively oriented orthonormal basis of TM, and such that around each p_k there is a partial parametrization in which $Y_j = Df(X_j)$ for some frame with singularity of order $m(p_i)$ at p.

THEOREM 34.4 (no proof). Moving frames with singularities always exist. Moreover, for any choice of such frame, the sum $\sum_i m(p_i)$ is the same. It agrees with a topological invariant of M, called the Euler characteristic $\chi(M)$.

The torus has Euler characteristic 0. More interestingly, the sphere has Euler characteristic 2.

Corollary 34.5 (Gauss-Bonnet theorem; sketch proof). For any compact surface $M \subset \mathbb{R}^3$, $\kappa_{gauss}^{tot} = 2\pi \cdot \chi(M)$.

Corollary 34.6. The Gauss map ν of a compact surface $M \subset \mathbb{R}^3$ satisfies $\chi(M) = 2\deg(\nu)$. In particular, $\chi(M)$ is always even.

There is also a direct topological proof of this, avoiding curvature. Note that there exist abstract compact surfaces (compact topological spaces locally homeomorphic to $\mathbb{R}^2$) with odd Euler characteristic, but those do not admit orientations, hence cannot be realized inside $\mathbb{R}^3$.

Corollary 34.7 (sketch proof). For any compact surface $M \subset \mathbb{R}^3$, $\int_M \|\kappa\| dvol_M \geq 4\pi$.

Lecture 35

The Euler characteristic $\chi(M)$ is defined for all sufficiently nice topological spaces, and in particular for compact hypersurfaces M of any dimension. It is an intrinsic quantity (a homeomorphism invariant). We do not give the definition here, except to mention that if M admits a moving frame without any singularities, then the Euler characteristic is zero.

THEOREM 35.1 (Hopf; no proof). Let $M \subset \mathbb{R}^{n+1}$ be a closed hypersurface of even dimension n, and $\nu : M \to S^n$ a Gauss map. Then $\deg(\nu) = \chi(M)/2$.

COROLLARY 35.2 (Generalized Gauss-Bonnet). In the same situation as above, $\kappa_{gauss}^{tot} = \chi(M)\mathrm{vol}(S^n)/2$.

No such result exists for odd n, which means that κ_{gauss}^{tot} is not intrinsic in those dimensions (it depends on how the hypersurface sits in $\mathbb{R}^{n+1}$).

DEFINITION 35.3. A *compact combinatorial surface* consists of a finite collection $\{P_i\}$ of flat convex polygons in $\mathbb{R}^3$, with the following properties: any two P_i are either disjoint or share a common edge; (ii) any edge of any given P_i belongs to precisely one other P_j, $j \neq i$.

We usually think of $M = \bigcup_i P_i$ as the surface. Write $\{E_j\}$ for the set of edges, and $\{V_k\}$ for the set of vertices. The combinatorial Gauss map assigns to each P_i a normal vector $\nu(P_i) \in S^2$, uniquely determined by the requirement that it should point outwards (if M is connected, this means pointing into the component of $\mathbb{R}^3 \setminus M$ which is not bounded). For each edge E_j we then get a great circle segment $\nu(E_j) \subset S^2$ connecting the normal vectors associated to its endpoints. Similarly, for each vertex V_k we get a "region" $\nu(V_k) \subset S^2$ whose boundaries are the great circle segments associated to the edges adjacent to each vertex. The combinatorial Gauss curvature is the spherical area

$$\kappa_{gauss}^{comb}(V_k) = \text{“area}(\nu(V_k))\text{”}.$$

This has to be approached with some care, since the "region" can have self-overlaps, and the area should be counted with sign. In the case of a convex vertex, one really gets the ordinary positive area. More generally, one can use some spherical trigonometry to get

$$\kappa_{gauss}^{comb}(V_k) = 2\pi - \sum \text{angles of corners adjacent to our vertex},$$

where the angles are counted with signs. Define the Euler characteristic to be $\chi(M) = \#\text{polygons} - \#\text{edges} + \#\text{vertices}$ (for a polygonal approximation of a smooth surface, this agrees with our previous definition). By applying spherical trigonometry, one obtains

THEOREM 35.4 (combinatorial Gauss-Bonnet; sketch proof). $\sum_k \kappa_{gauss}^{comb}(V_k) = 2\pi\chi(M)$.