Chapter 3

Inverse Function Theorem

(This lecture was given Thursday, September 16, 2004.)

3.1 Partial Derivatives

Definition 3.1.1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}^n$, then the limit*

$$D_i f(a) = \lim_{h \to 0} \frac{f(a^1, \ldots, a^i + h, \ldots, a^n) - f(a^1, \ldots, a^n)}{h} \tag{3.1}$$

*is called the i^{th} **partial derivative** of f at a, if the limit exists.*

Denote $D_j(D_i f(x))$ by $D_{i,j}(x)$. This is called a **second-order (mixed) partial derivative**. Then we have the following theorem (**equality of mixed partials**) which is given without proof. The proof is given later in Spivak, Problem 3-28.

Theorem 3.1.2. *If $D_{i,j} f$ and $D_{j,i} f$ are continuous in an open set containing a, then*

$$D_{i,j} f(a) = D_{j,i} f(a) \tag{3.2}$$

We also have the following theorem about partial derivatives and maxima and minima which follows directly from 1-variable calculus:

Theorem 3.1.3. *Let $A \subset \mathbb{R}^n$. If the maximum (or minimum) of $f : A \to \mathbb{R}$ occurs at a point a in the interior of A and $D_i f(a)$ exists, then $D_i f(a) = 0$.*

Proof: Let $g_i(x) = f(a^1, \ldots, x, \ldots, a^n)$. g_i has a maximum (or minimum) at a^i, and g_i is defined in an open interval containing a^i. Hence $0 = g_i'(a^i) = 0$.

The converse is not true: consider $f(x, y) = x^2 - y^2$. Then f has a minimum along the x-axis at 0, and a maximum along the y-axis at 0, but $(0, 0)$ is neither a relative minimum nor a relative maximum.

3.2 Derivatives

Theorem 3.2.1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then $D_j f^i(a)$ exists for $1 \le i \le m, 1 \le j \le n$ and $f'(a)$ is the m x n matrix $(D_j f^i(a))$.*

Proof: First consider $m = 1$, so $f : \mathbb{R}^n \to \mathbb{R}$. Define $h : \mathbb{R} \to \mathbb{R}^n$ by $h(x) = (a^1, \ldots, x, \ldots, a^n)$, with x in the j^{th} slot. Then $D_j f(a) = (f \circ h)'(a^j)$. Applying the chain rule,

$$\begin{aligned} (f \circ h)'(a^j) &= f'(a) \cdot h'(a^j) \\ &= f'(a) \cdot \begin{pmatrix} 0 \\ . \\ . \\ . \\ . \\ 1 \\ . \\ . \\ . \\ 0 \end{pmatrix} \end{aligned} \tag{3.3}$$

Thus $D_j f(a)$ exists and is the jth entry of the $1 \times n$ matrix $f'(a)$.

Spivak 2-3 (3) states that f is differentiable if and only if each f^i is. So the theorem holds for arbitrary m, since each f^i is differentiable and the ith row of $f'(a)$ is $(f^i)'(a)$.

The converse of this theorem – that if the partials exists, then the full derivative does – only holds if the partials are continuous.

Theorem 3.2.2. *If $f : \mathbb{R}^n \to \mathbb{R}^m$, then $Df(a)$ exists if all $D_j f(i)$ exist in an open set containing a and if each function $D_j f(i)$ is continuous at a. (In this case f is called* ***continuously differentiable****.)*

Proof.: As in the prior proof, it is sufficient to consider m = 1 (i.e., $f : \mathbb{R}^n \to \mathbb{R}$.)

$$\begin{aligned} f(a+h) - f(a) = \; & f(a^1 + h^1, a^2, \ldots, a^n) - f(a^1, \ldots, a^n) \\ & + f(a^1 + h^1, a^2 + h^2, a^3, \ldots, a^n) - f(a^1 + h^1, a^2, \ldots, a^n) \\ & + \ldots + f(a^1 + h^1, \ldots, a^n + h^n) \\ & - f(a^1 + h^1, \ldots, a^{n-1} + h^{n-1}, a^n). \end{aligned} \tag{3.4}$$

$D_1 f$ is the derivative of the function $g(x) = f(x, a^2, \ldots, a^n)$. Apply the mean-value theorem to g :

$$f(a^1 + h^1, a^2, \ldots, a^n) - f(a^1, \ldots, a^n) = h^1 \cdot D_1 f(b_1, a^2, \ldots, a^n). \tag{3.5}$$

for some b^1 between a^1 and $a^1 + h^1$. Similarly,

$$h^i \cdot D_i f(a^1 + h^1, \ldots, a^{i-1} + h^{i-1}, b_i, \ldots, a^n) = h^i D_i f(c_i) \tag{3.6}$$

for some c_i. Then

$$\begin{aligned}
&\lim_{h\to 0} \tfrac{|f(a+h)-f(a)-\sum_i D_i f(a)\cdot h^i|}{|h|} \\
&= \lim_{h\to 0} \tfrac{\sum_i [D_i f(c_i) - D_i f(a)\ \cdot h^i]}{|h|} \\
&\leq \lim_{h\to 0} \textstyle\sum_i |D_i f(c_i) - D_i f(a)| \cdot \tfrac{|h^i|}{|h|} \\
&\leq \lim_{h\to 0} \textstyle\sum_i |D_i f(c_i) - D_i f(a)| \\
&= 0
\end{aligned} \tag{3.7}$$

since $D_i f$ is continuous at 0.

Example 3. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be the function* $f(x, y) = xy/(\sqrt{x^2 + y^2}$ *if* $(x, y) \neq (0, 0)$ *and 0 otherwise (when* $(x, y) = (0, 0)$*). Find the partial derivatives at* $(0, 0)$ *and check if the function is differentiable there.*

3.3 The Inverse Function Theorem

(A sketch of the proof was given in class.)