Minimal Surfaces as Isotropic Curves in $\mathbb{C}^3$: Associated minimal surfaces and the Björling's problem

by

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Abstract

In this paper, we introduce minimal surfaces as isotropic curves in $\mathbb{C}^3$. Given such a isotropic curve, we can define the adjoint surface and the family of associate minimal surfaces to a minimal surface that is the real part of the isotropic curve. We study the behavior of asymptotic lines and curvature lines in a family of associate surfaces, specifically the asymptotic lines of a minimal surface are the curvature lines of its adjoint surface, and vice versa.

In the second part of the paper, we describe the Björling's problem. Given a real-analytic curve and a real-analytic vector field along the curve, Björling's problem is to find a minimal surface that includes the curve such that its unit normal field coincides with the given vector field. We shows that the Björling's problem always has a unique solution. We will use some examples to demonstrate how to construct minimal surface using the results from the Björling's problem. Some symmetry properties can be derived from the solution to the Björling's problem. For example, straight lines are lines of rotational symmetry, and planar geodesics are lines of mirror symmetry in a minimal surface. These results are useful in solving the Schwarzian chain problem, which is to find a minimal surface that spanned into a frame that consists of finitely many straight lines and planes.

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Chapter 1

Introduction

Minimal surface, such as soap film, has zero curvature at every point. It has attracted the attention for both mathematicians and natural scientists for different reasons. Mathematicians are interested in studying minimal surfaces that have certain properties, such as completedness and finite total curvature, while scientists are more inclined to periodic minimal surfaces observed in crystals or biosystems such as lipid bilayers.

Since a surface surrounded by a boundary is minimal if it is an area minimizer, early mathematician search minimal surface by solving the Lagrange variational problem, which leads to a partial differential equation for such area-minimizing surface. However, in Chapter 2, we will discuss how to describe minimal surfaces using the Weierstrass-Enneper representation, which can generate minimal surfaces given an arbitrary holomorphic "Weierstrass function". Such powerful tool allows us to choose different functions to generate minimal surfaces. The representation also gives rise to ideas such as isotropic curve, adjoint surfaces and associate family of surfaces. Curvature lines and asymptotic lines can also be expressed in the form of the Weierstrass function.

Chapter 3 is about "Björling's Problem", which is to find a minimal surface that coincides with a given real-analytic curve and a real-analytic normal field along the curve. We will show that it is simple to write down a unique solution that solves Björling's problem with Weierstrass-Enneper representation. In Chapter 4 we will study the Björling's problem further to show that straight lines and planar curves have interested symmetry properties in minimal surfaces. This provides the fundamentals to solve the Schwarzian chain problem, which is to find a minimal surface that is spanned into certain frame consisting finitely many straight lines and planes. These minimal surfaces are particularly interesting to scientists, since the "frame" can serve as a building block to bigger minimal surfaces using reflexive and translational symmetry.

Computer graphics gives visualization to minimal surface, and thus is a very useful tool in studying minimal surface. In the last chapter, we will briefly describe how to use computer programs such as Maple and Surface Evolver to study minimal surface.

Chapters 2 to 4 are based on [4], which provides a comprehensive account of minimal surface. The Maple procedures in Section 5.1 are discussed in [5]. For more information about the program Surface Evolver, reader are referred to the program manual [2].

Chapter 2

The Adjoint Surface and the Family of Associate Minimal Surfaces

The Weierstrass-Enneper representation provides a powerful tool to generate minimal surfaces. In this chapter, we will show that Weierstrass-Enneper representation is nothing but expressing a minimal surface X as the real part of some isotropic curve f in $\mathbb{C}^3$. Furthermore, the imaginary part of the isotropic curve is also a minimal surface, which is called the "adjoint surface" of X. We can also show that for every minimal surface X, there exists a one parameter associate family of minimal surfaces which includes X and its adjoint surface. We will finish this chapter by studying the behavior of asymptotic lines and curvature lines in the associate family, and some examples of minimal surfaces.

2.1 Weierstrass-Ennerper Representations

We start by stating the Weierstrass-Enneper representations of minimal surfaces.

Theorem 2.1.1 (Weierstrass-Ennerper Representation I). *For every noncon-*

stant minimal surface

$$X(w) = (x(w), y(w), z(w)), \quad w \in \Omega \subset \mathbb{C}$$

defined on a simply connected domain Ω*, there are a holomorphic function* μ *and a meromorphic function* ν *in* Ω *with* $\mu \not\equiv 0$, $\nu \not\equiv 0$ *such that* $\mu\nu^2$ *is holomorphic in* Ω*, and that*

$$\begin{array}{rcl} x(w) & = & \Re \int_{w_0}^{w} \frac{1}{2}\mu(1-\nu^2)\ d\zeta \\ y(w) & = & \Re \int_{w_0}^{w} \frac{i}{2}\mu(1+\nu^2)\ d\zeta \\ z(w) & = & \Re \int_{w_0}^{w} \mu\nu\ d\zeta \end{array} \tag{2.1}$$

holds for $w, w_0 \in \Omega$.

Conversely, two functions μ *and* ν *as above define by means of 2.1 a minimal surface* $X : \Omega \to \mathbb{R}^3$ *provided that* Ω *is simply connected.*

If we introduce a function

$$\mathcal{F}(w) = \frac{\mu(w)}{\nu'(w)} \tag{2.2}$$

then we have another representation of minimal surfaces:

Theorem 2.1.2 (Weierstrass-Ennerper Representation II). *Let* $\mathcal{F}(w)$ *be a holomorphic function in a simply connected domain* Ω *of* $\mathbb{C}$, $\mathcal{F} \not\equiv 0$ *and set*

$$\Phi(w) = ((1-w^2)\mathcal{F}(w), i(1+w^2)\mathcal{F}(w), 2w\mathcal{F}(w)). \tag{2.3}$$

Then

$$X(w) = \Re \int_{w_0}^{w} \Phi(w)\ dw, \quad w \in \Omega \tag{2.4}$$

defines a non-constant minimal surface $X : \Omega \to \mathbb{R}^3$.

2.2 Adjoint Surface

It is natural to ask what will happen if we consider the imaginary part of the arguments in the Weierstrass-Ennerper representation. Let $X(u, v) = (x(u, v), y(u, v), z(u, v)$

be a minimal surface defined on a simply connected domain $\Omega \in \mathbb{R} \cong \mathbb{C}$ with

$$\Delta X = 0 \tag{2.5}$$

$$\|X_u\|^2 = \|X_v\|^2, \quad \langle X_u, X_v \rangle = 0, \quad (u,v) \in \Omega. \tag{2.6}$$

Definition 2.2.1. *A surface $X^*(u,v) = (x^*(u,v), y^*(u,v), z^*(u,v))$ is said to be an* **adjoint surface** *to $X(u,v)$ on Ω if the following Cauchy-Riemann equations*

$$X_u = X_v^*, \quad X_v = -X_u^* \tag{2.7}$$

hold in Ω.

It is immediate that

$$\Delta X^* = 0, \quad \|X_u^*\|^2 = \|X_v^*\|^2, \quad \langle X_u^*, X_v^* \rangle = 0 \tag{2.8}$$

and thus X^* is also a minimal surface.

Consider a mapping

$$f(w) = X(u,v) + iX^*(u,v), \quad w = u + iv \in \Omega, \tag{2.9}$$

then f is a holomorphic mapping of Ω into $\mathbb{C}^3$ with components

$$\begin{aligned} f(w) = & \quad (f^1(w), f^2(w), f^3(w)) \\ f^j(w) = & \quad x^j(w) + i(x^*)^j(w) \end{aligned} \tag{2.10}$$

for $j = 1, 2, 3$.

If we define the complex derivative of f by

$$f' = \frac{df}{dw} = X_u + iX_u^* = X_u - iX_v, \tag{2.11}$$

then we have

$$\langle f', f' \rangle = \|X_u\|^2 - \|X_v\|^2 - 2i \langle X_u, X_v \rangle = 0 \tag{2.12}$$

We say that the holomorphic curve f is a **isotropic curve** since it satisfies $\langle f', f' \rangle = 0$.

We obtain the following result:

Proposition 2.2.2. *Let $X : \Omega \to \mathbb{R}^3$ be a minimal surface defined on a simply connected parameter domain $\Omega \subset \mathbb{C}$, and X^* be the adjoint surface of X, then the holomorphic curve $f : \Omega \to \mathbb{C}^3$ defined by*

$$f(w) = X(w) + iX^*(w), \quad w \in \Omega \tag{2.13}$$

is an isotropic curve.

Conversely, if $f : \Omega \to \mathbb{C}^3$ is an isotropic curve, then

$$X(u, v) = \Re f(w), \quad X^*(u, v) = \Im f(w), \quad , w = u + iv \tag{2.14}$$

defineds two minimal surface.

Corollary 2.2.3. *We say $X^*(u, v), w = u + iv \in \Omega$ is an adjoint surface to some $X(u, v), u + iv \in \Omega$ if there exists an isotropic curve $f : \Omega \to \mathbb{C}^3$ such that the Eq.(* 2.14) *is satisfied.*

2.3 Associate Minimal Surfaces

We have shown that minimal surfaces can ben represented by the real parts of isotropic curves in $\mathbb{C}^3$. We can now modify the isotropic curves slightly to give rise to a family of minimal surfaces that were discovered by Bonnet.

Given a isotropic curve $f(w) = X(w) + iX^*(w)$ where $w = u + iv \in \Omega$ and $\langle f'(w), f'(w) \rangle = 0$, we can define another holomorphic function

$$g(w, \theta) = e^{-i\theta} f(w), \quad w \in \Omega. \tag{2.15}$$

It is easy to see that $g(w,\theta)$ is an isotropic curve, and

$$Z(w,\theta) = \Re e^{-i\theta} f(w) = X(w)\cos\theta + X^*(w)\sin\theta \tag{2.16}$$

defines a one-parameter family of minimal surfaces such that

$$Z(w,0) = X(w), \quad Z(w,\frac{\pi}{2}) = X^*(w). \tag{2.17}$$

Since

$$Z_u = X_u\cos\theta - X_v\sin\theta, \quad Z_v = X_v\cos\theta + X_u\sin\theta, \tag{2.18}$$

we have

$$\|Z_u\|^2 = \|Z_v\|^2 = \|X_u\|^2 = \|X_v\|^2 \tag{2.19}$$

and thus

$$\langle dZ(\cdot,\theta), dZ(\cdot,\theta)\rangle = \langle dX, dX\rangle\,. \tag{2.20}$$

Therefore, all associate minimal surfaces to a given minimal surface have the same first fundamental form, i.e. they are isometric to each other.

If we denote $f' = \Phi = (\Phi_1, \Phi_2, \Phi_3)$, then we have from Eq. (**??**) that

$$Z(w,\theta) = \Re\int_{w_0}^{w} e^{-i\theta}\Phi(w)dw. \tag{2.21}$$

In the Weierstrass-Ennerper representation, we then have a “Weierstrass function“, $\tilde{\mathcal{F}}(w,\theta)$, for the family given by

$$\tilde{\mathcal{F}}(w,\theta) = e^{-i\theta}\mathcal{F}(w). \tag{2.22}$$

In other words, if $\mathcal{F}(w)$ is the Weierstrass function to a minimal surface X, then $i\mathcal{F}(w)$ is the Weierstrass function to the corresponding adjoint surface.

2.4 Behavior of Asymptotic Lines and Curvature Lines in a Family of Associate Minimal Surfaces

Let $\omega(t) = (\alpha(t), \beta(t))$ be a C^1 curve in Ω, $t \in [-\epsilon, \epsilon]$, then

- $\omega(t)$ is an asymptotic line of X if and only if

$$e\dot{\alpha}^2 + 2f\dot{\alpha}\dot{\beta} + g\dot{\beta}^2 = 0. \tag{2.23}$$

- $\omega(t)$ is a line of curvature of X if and only if

$$(eF - fE)\dot{\alpha}^2 + (Eg - Ge)\dot{\alpha}\dot{\beta} + (gF - fG)\dot{\beta}^2 = 0 \tag{2.24}$$

In an isothermal parametrization, we have $e = -g$ and $E = G$, thus the asymptotic lines are described by

$$e(\dot{\alpha}^2 - \dot{\beta}^2) + 2f\dot{\alpha}\dot{\beta} = 0 \tag{2.25}$$

and the curvature lines are characterized by

$$f(\dot{\alpha}^2 - \dot{\beta}^2) - 2e\dot{\alpha}\dot{\beta} = 0 \tag{2.26}$$

If we introduce a complex valued quadratic form $\Xi(\dot{w})$:

$$\Xi(\dot{w}) = l(w)(\dot{\alpha} + i\dot{\beta})^2, \quad l(w) = e(w) - if(w) = \langle f''(w), N(w)\rangle\,, \tag{2.27}$$

then the asymptotic lines and the curvature lines are given by

$$\Re\{\Xi(\dot{w})\} = 0 \quad and \quad \Im\{\Xi(\dot{w})\} = 0 \tag{2.28}$$

or equivalently

$$\Re\{l(w)(dw)^2\} = 0 \quad and \quad \Im\{l(w)(dw)^2\} = 0 \tag{2.29}$$

Now if we consider a holomorphic function

$$l(\theta) = e(\theta) - if(\theta) = \langle g''(\cdot, \theta), N \rangle \tag{2.30}$$

which characterizes the asymptotic lines and the curvature lines of the associate minimal surface $Z(\cdot, \theta)$. Since $g''(w, \theta) = e^{-i\theta} f''(w)$, we have

$$l(\theta) = e^{-i\theta} l(0) = [e \cos\theta - f \sin\theta] - i[e \sin\theta + f \cos\theta] \tag{2.31}$$

where $l(0) = l = e - if$ is the characteristic function for $X = Z(\cdot, 0)$. If we set

$$\begin{aligned} \xi \quad &= e(\dot{\alpha}^2 - \dot{\beta}^2) + 2f\dot{\alpha}\dot{\beta}, \\ \eta \quad &= -f(\dot{\alpha}^2 \ \dot{\beta}^2) + 2e\dot{\alpha}\dot{\beta}, \end{aligned} \tag{2.32}$$

then we have

$$\begin{aligned} l(0)(\dot{\alpha} + i\dot{\beta})^2 \quad &= \xi + i\eta \\ l(\tfrac{\pi}{2})(\dot{\alpha} + i\dot{\beta})^2 \quad &= \eta - i\xi \end{aligned} \tag{2.33}$$

Since $X = Z(\cdot, 0)$ and $X^* = Z(\cdot, \frac{\pi}{2})$, we showed that the asymptotic lines of X are the curvature lines of X^*, and the curvature lines of X are asymptotic lines of X^*.

We can now express the results of this section in terms of the function $\mathcal{F}(w)$ in the Weierstrass-Ennerper Representation. In doing so, we would like to express the characteristic function $l(w) = \langle N, f'' \rangle = e - if$ in terms of the functions μ, ν and $\mathcal{F}$ in the representation.

Consider an isotropic curve $f : \Omega \to \mathbb{C}^3$ that is related to a minimal surface $X(w), w \in \Omega$ by the formula

$$X(w) = \Re\{f(w)\}. \tag{2.34}$$

We obtain from the first Weierstrass-Ennerper representation that

$$f' = X_u - iX_v = \left(\frac{1}{2}\mu(1 - \nu^2), \frac{i}{2}\mu(1 + \nu^2), \mu\nu \right). \tag{2.35}$$

$$\begin{tikzcd}
\Omega \arrow[r] \arrow[rrr, bend left, "\nu"] & X \arrow[r] & N \arrow[r, "\sigma"] & \Omega^*
\end{tikzcd}$$

Figure 2-1: The map ν

If we set $g = (-\nu, i\nu, 1)$, we obtain

$$f'' = X_{uu} - iX_{uv} = \frac{\mu'}{\mu} f' + \mu\nu' g \tag{2.36}$$

From Eq. (2.11), let $\Phi = (\Phi_1, \Phi_2, \Phi_3) = f'$, then we have

$$X_u = \Re\Phi, \quad X_v = \Im\Phi. \tag{2.37}$$

Then we can obtain

$$X_u \wedge X_v = \Im(\Phi_2\bar{\Phi}_3, \Phi_3\bar{\Phi}_1, \Phi_1\bar{\Phi}_2) \tag{2.38}$$

and consequently

$$N = \frac{2}{\|\Phi\|^2}\Im(\Phi_2\bar{\Phi}_3, \Phi_3\bar{\Phi}_1, \Phi_1\bar{\Phi}_2) = \frac{1}{1+\|\nu\|^2}(2\Re\nu, 2\Im\nu, \|\nu\|^2 - 1). \tag{2.39}$$

It is easy to see that $\langle N, g\rangle = -1$ and $\langle N, f'\rangle = 0$. Combining Eqs. (2.36) and (2.39), we have

$$l = e - if = \langle N, f''\rangle = -\mu\nu'. \tag{2.40}$$

By comparing the representation formulas in Eqs. (2.1) and (2.3), we note that one representation can go over into another one if we set $\mu(w) = 2\mathcal{F}(w)$ and $\nu(w) = w$ and we then have three ways to characterize asymptotic lines and curvature lines, as shown in Table 2.1.

Table 2.1: Characterization of Asymptotic Lines and Curvature Lines in Different Representation of Minimal Surfaces

Asymptotic Lines	Curvature Lines	Remarks
$\Re l(w)(dw)^2 = 0$	$\Im l(w)(dw)^2 = 0$	$l = e - if$
$\Re\{\mu(\gamma)\nu'(\gamma)\dot{\gamma}^2\} = 0$	$\Im\{\mu(\gamma)\nu'(\gamma)\dot{\gamma}^2\} = 0$	$\gamma(t) = \alpha(t) + i\beta(t)$
$\Re\mathcal{F}(w)(dw)^2 = 0$	$\Im\mathcal{F}(w)(dw)^2 = 0$	

2.5 Catenoid and Helicoid

Catenoid is obtained by rotating a catenary about the z-axis in $\mathbb{R}^3$. A catenary has the form

$$x = \alpha \cosh\left(\frac{z - z_0}{\alpha}\right) \tag{2.41}$$

for $z \in \mathbb{R}$, and z_0, α are arbitrary constants with $\alpha \neq 0$. If we set $z_0 = 0$, then the catenoid $X(u, v)$ can be parametrized by

$$\begin{aligned} x(u,v) &= \alpha \cosh u \cos v \\ y(u,v) &= -\alpha \cosh u, \sin v \\ z(u,v) &= \alpha u \end{aligned} \tag{2.42}$$

with $-\infty < u < \infty$ and $0 \leq v < 2\pi$. By the following formulas

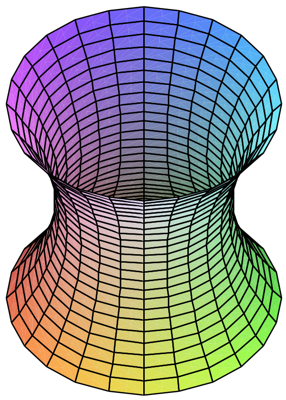
$$\begin{aligned} \cosh(u + iv) &= \cosh u \cos v + i \sinh u \sin v \\ \sinh(u + iv) &= \sinh u \cos v + i \cosh u \sin v \end{aligned} \tag{2.43}$$

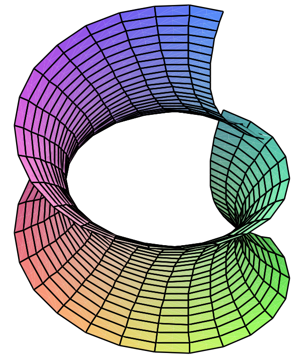
we find that the catenoid can be written as $X(w) = \Re f(w)$ if we set the isotropic curve f to be

$$f(w) = (\alpha \cosh w, \alpha i \sinh w, \alpha w) \tag{2.44}$$

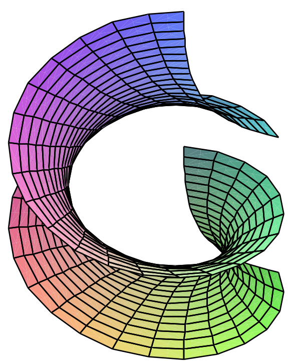
In order to find the Weierstrass function $\mathcal{F}(\omega)$ of the catenoid such that

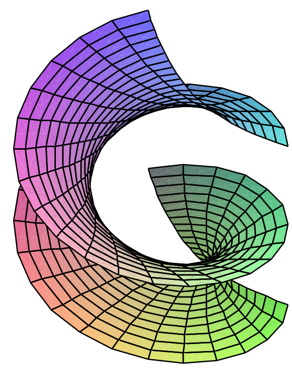
$$\begin{aligned} x &= \alpha + \Re \textstyle\int_1^\omega (1 - \omega^2)\mathcal{F}(\omega)\ d\omega \\ y &= \Re \textstyle\int_1^\omega i(1 + \omega^2)\mathcal{F}(\omega)\ d\omega \\ z &= \Re \textstyle\int_1^\omega 2\omega\mathcal{F}(\omega)\ d\omega \end{aligned} \tag{2.45}$$



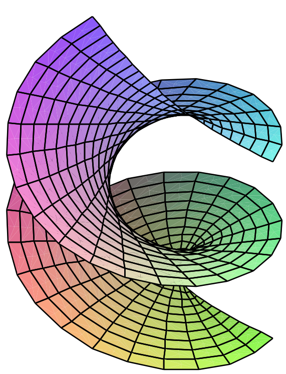


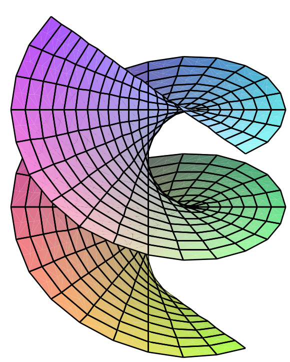
(a) (b)





(c) (d)





(e) (f)

Figure 2-2: The bending process from catenoid to helicoid we introduce a new variable $\omega = e^{-w} = e^{-u-iv}$. We know that ω can be represented in polar form $\omega = re^{i\theta}$. Then $\log \omega = \log r + i\theta = -u - iv$. Thus the representation of catenoid in Eq. (2.42) can be written as

$$\begin{aligned} x &= \frac{\alpha}{2}\left(\frac{1}{r}+r\right)\cos\theta &&= \Re\frac{\alpha}{2}\left(\frac{1}{\omega}+\omega\right) \\ y &= \frac{\alpha}{2}\left(\frac{1}{r}+r\right)\sin\theta &&= \Re\frac{i\alpha}{2}\left(\frac{1}{\omega}-\omega\right) \\ z &= -\alpha\log r &&= -\Re\alpha\log\omega. \end{aligned} \tag{2.46}$$

Comparing the expression for z in Eq. (2.42) and (2.46), we have

$$-\frac{d}{d\omega}\alpha\log\omega = 2\omega\mathcal{F}(\omega) \quad \Rightarrow \quad \mathcal{F}(\omega) = -\frac{\alpha}{2\omega^2} \tag{2.47}$$

From Eq. (2.44), we obtain the expression for the adjoint surface $X^*(w) = \Im f(w)$ of the catenoid:

$$\begin{aligned} x^*(u,v) &= \alpha\sinh u\sin v \\ y^*(u,v) &= \alpha\sinh u\cos v \\ z^*(u,v) &= \alpha v. \end{aligned} \tag{2.48}$$

If we write

$$X^* = \alpha Y(v) + \sinh u Z(v) \tag{2.49}$$

with

$$Y(v) = (0,0,v), \quad Z(v) = (\sin v, \cos v, 0), \tag{2.50}$$

we see that for every $v \in \mathbb{R}$, the curve $X^*(\cdot, v)$ is a straight line which meets the z-axis perpendicularly. If we fix $u \neq 0$, then $X^*(u,\cdot)$ describes a helix. Therefore the surface X^* is called a **helicoid**, the adjoint of the catenoid X.

2.6 Scherk's Second Surface

Scherk's second surfaces $Y(\omega)$ are defined by

$$\begin{aligned} x &= \alpha + \Re \int_1^\omega (1-\omega^2)\mathcal{F}(\omega)\ d\omega \\ y &= \Re \int_1^\omega i(1+\omega^2)\mathcal{F}(\omega)\ d\omega \\ z &= \Re \int_1^\omega 2\omega\mathcal{F}(\omega)\ d\omega \end{aligned} \tag{2.51}$$

with the Weierstrass function

$$\mathcal{F}(\omega) = \frac{-(\alpha - i\beta)}{2\omega^2} \tag{2.52}$$

where $\alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 \neq 0$. Thus, Scherk's second surface is an associate surface of the catenoid. Specifically, for $\alpha = 0$ or $\beta = 0$, we obtain a helicoid or a catenoid respectively. If we represent ω by $e^{-w} = e^{-u-iv}$, then the Scherk's second surface can be expressed in a parameterized form

$$\begin{aligned} x &= \alpha \cosh u \cos v + \beta \sinh u \sin v \\ y &= -\alpha \cosh u \sin v + \beta \sinh u \cos v \\ z &= \alpha u + \beta v + \gamma. \end{aligned} \tag{2.53}$$

If we represent a catenoid by

$$X^{\mathrm{cat}}(w) = (\cosh u \cos v, -\cosh u \sin v, u) \tag{2.54}$$

and a helicoid

$$X^{\mathrm{hel}}(w) = (\sinh u \sin v, \sinh u \cos v, v) \tag{2.55}$$

with $\gamma = 0$, then we can write

$$X(w) = c[\cos\theta X^{\mathrm{cat}}(w) + \sin\theta X^{\mathrm{hel}}(w)] \tag{2.56}$$

where $\alpha = c\cos\theta, \beta = c\sin\theta$ and $c = \sqrt{\alpha^2 + \beta^2}$. This clearly shows that the Scherk's second surface is nothing but an associate surface of the catenoid.

2.7 The Enneper Surface

A minimal surface $X(w), w \in \mathbb{C}$ with a Weierstrass representation

$$\mathcal{F}(w) \equiv 1 \tag{2.57}$$

is called an **Enneper Surface**. Using the Weierstrass-Enneper representation, we have

$$X(w) = \Re\left(w - \frac{w^3}{3}, iw + \frac{iw^3}{3}, w^2\right). \tag{2.58}$$

Thus the components of $X(u, v)$ are given by

$$\begin{aligned} x &= u - \tfrac{1}{3}u^3 + uv^2 \\ y &= -v - u^2 v + \tfrac{1}{3}v^3 \\ z &= u^2 - v^2 \end{aligned} \tag{2.59}$$

where $w = u + iv \in \mathbb{C}$.

The associate surfaces of the Enneper surface, $Z(w, \theta)$, is given by

$$Z(w,\theta) = \Re\left\{e^{-i\theta}\left(w - \frac{w^3}{3}\right), ie^{-i\theta}\left(w + \frac{w^3}{3}\right), e^{-i\theta}w^2\right\}. \tag{2.60}$$

We will show that the trace of the Enneper surface is congruent to the traces of its associate surfaces. Let us define a set of new cartesian coordinates ξ, η, z which is a rotation of the set x, y, z about the z-axis with an angle $-\frac{\theta}{2}$:

$$\xi + i\eta = e^{-\frac{i\theta}{2}}(x + iy). \tag{2.61}$$

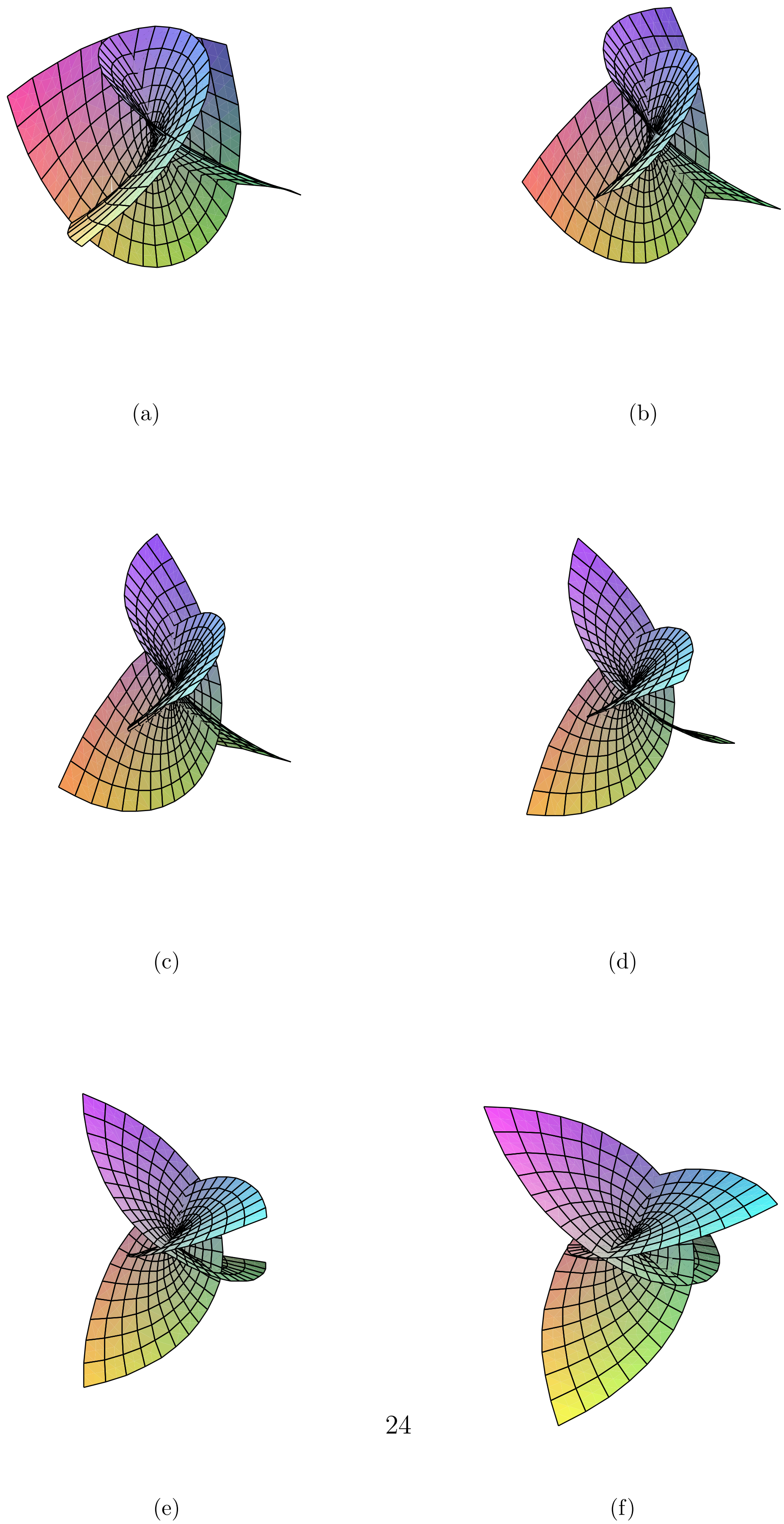


Figure 2-3: The Enneper Surface and it's associate minimal surface

Thus the new coordinates of the associate surface will be

$$\begin{aligned} \eta(w) + i\eta(w) = \quad & e^{-i\theta/2}[\Re(e^{-i\theta}w) + i\Re(ie^{-i\theta}w)] \\ & + e^{-i\theta/2}\left[\Re\left(-\tfrac{1}{3}e^{-i\theta}w^3\right) + i\Re\left(ie^{-i\theta}\tfrac{w^3}{3}\right)\right] \end{aligned} \tag{2.62}$$

If we denote $\zeta = e^{-i\theta/2}w$, then Eq. (2.62) becomes

$$\xi + i\eta = \Re(\zeta - \frac{1}{3}\zeta^3) + i\Re i(\eta + \frac{1}{3}\eta^3). \tag{2.63}$$

Rearranging, we have

$$\begin{aligned} \xi \quad & = \Re(\zeta - \tfrac{1}{3}\zeta^3) \\ \eta \quad & = \Re i(\zeta + \tfrac{1}{3}\zeta^3) \\ z \quad & = \Re\zeta^2 \end{aligned} \tag{2.64}$$

which is identical to Eq. (2.58) with a change of variables.

Chapter 3

Björling's Problem

Given a real-analytic curve $c : I \to \mathbb{R}^3$ with $\dot{c}(t) \neq 0$, and a real-analytic vector field $n : I \to \mathbb{R}^3$ along c such that $\|n(t)\| \equiv 1$ and $\langle \dot{c}(t), n(t) \rangle \equiv 0$, the Björling's problem concerns about whether one can find a minimal surface $X : \Omega \to \mathbb{R}^3$ with $I \subset \Omega$ such that

- $X(u, 0) = c(u)$
- $N(u, 0) = n(u)$

3.1 Solution to the Björling's Problem

Theorem 3.1.1. *For any prescribed real-anlytic strip $S = \{c(t), n(t) : t \in I\}$, the corresponding Björling problem has exactly one solution $X(u, v)$ given by*

$$X(u, v) = \Re \left\{ c(w) - i \int_{w_0}^{w} n(w) \wedge \dot{c}(w) \; dw \right\} \tag{3.1}$$

where $w = u + iv \in \Omega$, $u_0 \in I$, and Ω a simply connected domain with $I \subset \Omega$ in which the power series expansions of both c and n are convergent.

In Eq. (3.1), we determine holomorphic extensions $c(u + iv)$ and $n(u + iv)$ of the real-analytic functions $c(t)$ and $n(t), t \in I$, to a suitable simply connected domain Ω

that contains I and then determine the line integral

$$\int_{u_0}^{w} n(w) \wedge c'(w)\ dw$$

where $c'(w)$ is the complex derivative of the holomorphic function $c(w)$.

Proof. Let us first prove the uniqueness of the solution. Suppose $X(u,v)$ is a solution of Björling's problem, defined in the simply connected domain Ω, and let $X^* : \Omega \to \mathbb{R}^3$ be an adjoint surface to X with $X^*(u_0, 0) = 0$ and $u_0 \in I$. Then the function

$$f(w) = X(u,v) + iX^*(u,v), \quad w = u + iv \in \Omega \tag{3.2}$$

is an isotropic curve with $X = \Re f$, and

$$f' = X_u + ix_u^* = X_u - iX_v. \tag{3.3}$$

Since $X_v = N \wedge X_u$, we have

$$f' = X_u - iN \wedge X_u \tag{3.4}$$

which means

$$f'(u) = \dot{c}(u) - in(u) \wedge \dot{c}(u). \tag{3.5}$$

Integrating both sides,

$$f(u) = c(u) - i\int_{u_0}^{u} n(t) \wedge \dot{c}(t)\ dt \tag{3.6}$$

for all $u \in I$. Since both sides are holomorphic function of w, then both sides must agree on the whole plane, thus

$$f(w) = c(w) - i\int_{u_0}^{w} n(w) \wedge \dot{c}(w)\ dw \tag{3.7}$$

for $w \in \Omega$. Therefore any possible solution to X must be in the form of Eq. (3.1), which show the uniqueness.

Now we will show that Eq. (3.1) is the solution to Björling's problem. Consider the holomorphic curve $f : \Omega \to \mathbb{C}^3$ defined in Eq. (3.7). For $w \in I$,

$$\Re f'(w) = \dot{c}(w), \quad \Im f'(w) = -n(w) \wedge \dot{c}(w). \tag{3.8}$$

Since $\dot{c}(w)$ is orthogonal to $\dot{c}(w) \wedge n(w)$, we have

$$\langle f'(w), f'(w) \rangle = 0$$

for all $w \in I$, and thus for all $w \in \Omega$. Therefore $X(u,v) = \Re f(w)$ is a minimal surface. Since $c(w), n(w)$ and $c'(w)$ are real for $w \in I$,

$$X(u,0) = \Re f(u) = c(u) \tag{3.9}$$

and

$$X_u(u,0) - iX_v(u,0) = f'(u) = \dot{c}(u) - in(u) \wedge \dot{c}(u) \tag{3.10}$$

for $u \in I$. Moreover,

$$X_v(u,0) = N(u,0) \wedge X_u(u,0) \tag{3.11}$$

Since

$$\langle X_u(u,0), X_v(u,0) \rangle = 0, \quad \langle n(u), \dot{c}(u) \rangle = 0, \quad \|N(u,0)\| = \|n(u)\| = 1, \tag{3.12}$$

we have

$$N(u,0) = n(u) \tag{3.13}$$

□

Proposition 3.1.2. *If the curve $c(t)$ is contained in the xz-plane, $c(t) = (\xi(t), 0, \zeta(t))$, with a unit normal vector field $N(t)$, then the solution to Björling's problem is given* *by*

$$X(u,v) = \left(\Re\xi(w), \Im\int_0^w \sqrt{\xi'(w)^2 + \zeta'(w)^2}\ dw, \Re\zeta(w)\right) \tag{3.14}$$

Proof. Since $c'(t) = (\xi'(t), 0, \zeta'(t))$, the unit normal is

$$N = \frac{(-\zeta', 0, \xi')}{\sqrt{\xi'^2 + \zeta'^2}} \tag{3.15}$$

Then $N \wedge c' = (0, \sqrt{\xi'^2 + \zeta'^2}, 0)$. Hence

$$\Re\left(c - i\int N \wedge c'\right) = \left(\Re\xi, \Im\int \sqrt{\xi'^2 + \zeta'^2}\ dw, \Re\zeta\right). \tag{3.16}$$

□

3.2 Catalan Surface

A Catalan surface is obtained by solving the Björling's problem to find a minimal surface that contains a cycloid such that the surface normal conincides with the cycloid's principal normal vector. A cycloid is the curve generated by a point P on the circumference of a circle with center C rolling along a straight line.

A cycloid on the $x - z$ plane can be written as

$$c(t) = (1 - \cos t, 0, t - \sin t). \tag{3.17}$$

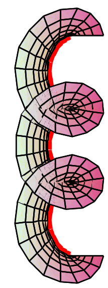
Let the Catalan surface be $X(u,v)$. Then by Eq. (3.14), we have

$$\begin{aligned} x(u,v) &= \Re(1 - \cos z) & z(u,v) &= \Re(z - \sin z) \\ &= 1 - \cos u \cosh v & &= u - \sin u \cosh v \end{aligned} \tag{3.18}$$

To compute $y(u,v)$, we have $\dot{c}(w) = (\sin w, 0, 1 - \cos w)$ and

$$\sin^2 w + (1 - \cos w)^2 = 4\sin^2\frac{w}{2}. \tag{3.19}$$

(a) A cycloid



(b) A cycloid contained in a Catalan's surface

Figure 3-1: A Catalan's surface solve the Björling's problem of a cycloid

Thus

$$\int \sqrt{\sin^2 w + (1-\cos w)^2}\, dw = -4\cos\frac{w}{2} \tag{3.20}$$

Since $\Im(-4\cos\frac{w}{2}) = 4\sin\frac{u}{2}\sinh v2$, we come to the espression of the Catalan's surface:

$$X(u,v) = \Big(1-\cos u\cosh v, 4\sin\frac{u}{2}\sinh v2, u - \sin u\cosh v\Big) \tag{3.21}$$

From Eq. (3.21), we infer that our Catalan surface is

$$X(u,v) = \Re f(w) \tag{3.22}$$

where the isotropic curve $f(w)$ is given by

$$f(w) = \left(1-\cosh(iw), 4i\cosh\left(\frac{iw}{2}\right), w + i\sinh(iw)\right) \tag{3.23}$$

Therefore, the adjoint surface $X^*(u,v)$ of Catalan's surface is given by

$$\begin{aligned} x^* &= \sin u \sinh v \\ y^* &= 4\cos\tfrac{u}{2}\cosh\tfrac{v}{2} \\ z^* &= v - \cos u \sinh v. \end{aligned} \tag{3.24}$$

3.3 Henneberg Surface

The Henneberg surface is obtained by solving the Björling's problem for the Neil's parabola

$$2x^3 = 9z^2 \tag{3.25}$$

which can be parameterized by

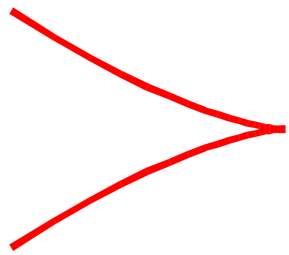
$$\begin{aligned} c(t) &= (x(t), 0, z(t)) \\ &= (\cosh(2t), 0, -\sinh t + \tfrac{1}{3}\sinh(3t)). \end{aligned} \tag{3.26}$$

By using Eq. (3.14), we have

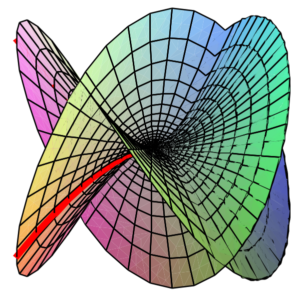
$$\begin{aligned} x(u,v) &= \Re(\cosh(2w) - 1) & \qquad z(u,v) &= \Re(-\sinh w + \tfrac{1}{3}\sinh(3w)) \\ &= -1 + \cosh 2u \cos 2v & &= \sinh u \cos v + \tfrac{1}{3}\sinh 3u \cos 3v \end{aligned} \tag{3.27}$$

To obtain $y(u,v)$, first note that $\dot{c}(w) = (2\sinh 2w, 0, -\cosh w + \cosh 3w)$, thus

$$\begin{aligned} & \Im \int \sqrt{\xi'(w)^2 + \zeta'(w)^2}\ dw \\ = \ & \Im \int \sqrt{2\cosh^2 2w - 4 + \cosh^2 w - 2\cosh w \cosh 3w + \cosh^2 3w}\ dw \\ = \ & \Im \int \sinh 3w + \sinh w \ dw \\ = \ & \Im \left(\tfrac{1}{3}\cosh 3w + \cosh w\right) \\ = \ & \tfrac{1}{3}\sinh 3u \sin 3v + \sinh u \sin v \end{aligned} \tag{3.28}$$



(a) Neil's parabola



(b) A Neil's parabola contained in a Henneberg surface

Figure 3-2: A Henneberg surface solve the Björling's problem of a Neil's parabola

Therefore, the Henneberg surface has a representation

$$\begin{aligned} x &= -1 + \cosh 2u \cos 2v \\ y &= -\sinh u \sin v - \tfrac{1}{3} \sinh 3u \sin 3v \\ z &= \sinh u \cos v + \tfrac{1}{3} \sinh 3u \cos 3v. \end{aligned} \tag{3.29}$$

The isotropic curve $f(w)$ that gives rise to the Henneberg surface X(u,v) with

$$X(u, v) = \Re f(w)$$

is given by

$$f(w) = \left(-1 + \cosh 2w, i \cosh w + \frac{i}{3} \cosh 3w, \sinh w + \frac{1}{3} \sinh 3w \right). \tag{3.30}$$

Thus the adjoint surface X^* to X is given by

$$\begin{aligned} x^* &= \sinh 2u \sin 2v \\ y^* &= \cosh u \cos v + \tfrac{1}{3} \cosh 3u \sin 3v \\ z^* &= -\cosh u \sin v + \tfrac{1}{3} \cosh 3u \sin 3v. \end{aligned} \tag{3.31}$$

We would like to show that the Henneberg surface is not orientable. Consider the Eq. (3.29), we can see that

$$\begin{aligned} X(u,v) &= X(-u, v+\pi) \\ X_u(u,v) &= -X_u(-u, v+\pi) \\ X_v(u,v) &= X_v(-u, v+\pi) \end{aligned} \tag{3.32}$$

for all $u + iv \in \mathbb{C}$. The curve

$$\omega(t) = (2t-1, \pi(t - \frac{1}{4})), \quad 0 \le t \le 1 \tag{3.33}$$

joins some point (u,v) with $(-u, v+\pi)$. Then the curve $\xi(t) = X(\omega(t)), 0 \le t \le 1$ is a closed regular loop on Henneberg's surface. But since $N(\omega(0)) = -N(\omega(1))$, so we will return to the initial point if we move around the loop $\xi(t)$, but the surface normal $N(\omega(0))$ will change to its opposite. This shows that the Henneberg's surface is a one-sided surface.

Chapter 4

More Applications of Björling's Problem

The solution to Björling's problem can be used to show some symmetry properties of minimal surface. In this chapter, we will show that straight lines and planar curves are lines of rotational symmetry and or mirror symmetry respectively. We will also discuss briefly about the Schwarzian chain problem, which is to find a minimal surface that are bounded by straight arcs and planar curves.

4.1 Symmetry of Minimal Surface

The solution to the Björling's problem also reveals some of the symmetry principles of minimal surfaces. We start with the following corollory of Theorem 3.1.1:

Corollary 4.1.1. *Let $X(u,v)$ be the solution to the Björling's problem given by Eq. (3.1), then we have*

$$X(u,-v) = \Re\left\{c(w) + i\int_{w_0}^{w} n(w)\wedge \dot{c}(w)\ dw\right\}, \quad w = u+iv. \tag{4.1}$$

Proof. Let $\tilde{X}(u,v) = X(u,-v)$. Then $\tilde{X}(u,v)$ is a minimal surface with the normal

$\tilde{N}(u,v) = -N(u,-v)$. Therefore, $\tilde{X}(u,v)$ solves the Björling's problem for the strip

$$\tilde{S} = \{(c(t), -n(t)) : t \in I\}$$

and thus

$$X(u,-v) = \Re\left\{c(w) + i\int_{w_0}^{w} n(w) \wedge \dot{c}(w)\right\}.$$

□

Proposition 4.1.2. *Let $X(u,v) = (x(u,v), y(u,v), z(u,v)), w = u + iv \in \Omega$ be a nonconstant minimal surface whose domain of definition Ω contains some interval I that lies on the real axis.*

1. *If, for all $u \in I$, the points $X(u,0)$ are contained in the x-axis, then we have*

$$\begin{aligned} x(u,-v) &= x(u,v) \\ y(u,-v) &= -y(u,v) \\ z(u,-v) &= -z(u,v) \end{aligned} \tag{4.2}$$

2. *If the curve $\Sigma = \{X(u,0) : u \in I\}$ is contained in the xy-plane E, and if the surface X intersects E orthogonally at Σ, i.e. the normal field $N(u,0)$ is also contained in the xy-plane, then it follows that*

$$\begin{aligned} x(u,-v) &= x(u,v) \\ y(u,-v) &= y(u,v) \\ z(u,-v) &= -z(u,v). \end{aligned} \tag{4.3}$$

   We call such Σ a **plane curve***.*

Proof. 1. Let $c(u) = (c^1(u), 0, 0), n(u) = (0, n^2(u), n^3(u))$, then

$$n(u) \wedge \dot{c}(u) = (0, \dot{c}^1(u)n^3(u), -\dot{c}^1(u)n^2(u)).$$

2. We have

$$c(u) = (c^1(u), c^2(u), 0), \quad n(u) = (n^1(u), n^2(u), 0), \tag{4.4}$$

and thus

$$n(u) \wedge \dot{c}(u) = (0, 0, n^1(u)\dot{c}^2(u), -n^2(u)\dot{c}^1(u)). \tag{4.5}$$

$\square$

Thus, we have proved the following theorem:

Theorem 4.1.3 (H. A. Schwarz). *1. Every straight line contained in a minimal surface is an axis of symmetry of the surface.*

2. If a minimal surface intersects some plane E perpendicularly, then E is a plane of symmetry of the surface.

$\square$

4.2 Straight Lines and Plane Curves in Minimal Surfaces

In this section we will prove that straight lines and plane curves are asymptotic lines and curvature lines respectively on a regular surface.

Theorem 4.2.1. *Let $X : \Omega \subset \mathbb{C} \to \mathbb{R}^3$ be a C^3 regular surface, $\omega(t), t \in I \subset \mathbb{R}$ be a C^3-curve in Ω, $w = u + iv \in \Omega$. Then $c(t) = X(\omega(t))$ is a regular curve in the surface X, and*

1. *The curve c is both a geodesic and asymptotic line if and only if it is a straight line.*

2. *Let c be a geodesic. Then c is a line of curvature if and only if it is a planar curve.*

Proof. Let the parameter t coincides with the arc length s. Denote $\{\mathcal{T}(s), \mathcal{S}(s), \mathcal{R}(s)$ be a moving orthogonal frame along $c(s)$, where $\mathcal{T}(s)$ is the tangent vector, $\mathcal{R}(s) = N(\omega(s))$ is the surface normal, and $\mathcal{S}(s) = \mathcal{R}(s) \wedge \mathcal{T}(s)$ is the side normal. MOrevoer, let $\mathcal{N}(s)$ be the principal normal and $\mathcal{B}(s) = \mathcal{T}(s) \wedge \mathcal{N}(s)$ be the binormal vector along the curve $c(s)$.

The vector $\dot{\mathcal{T}}$ satisfies the relation

$$\dot{\mathcal{T}} = \kappa_g \mathcal{S} + \kappa_n \mathcal{R} \tag{4.6}$$

which is explained in the Appendix A.

1. If c is both a geodesic and asymptotic line, then $\kappa_g = 0$ and $\kappa_n = 0$. Then Eq. (A.2) gives $\dot{\mathcal{T}} = 0$, which means that $c(s)$ is a straight line. Conversely, if $c(s)$ is a straight line, then $\dot{\mathcal{T}}(s) \equiv 0$, which implies that $\kappa_g(s) \equiv 0$ and $\kappa_n(s) \equiv 0$.

2. If $c(s)$ is a geodesic, then $\kappa_g = 0$ and $\dot{\mathcal{T}}$ will be parallel to $\mathcal{R}$. Since $\dot{\mathcal{T}} = \kappa \mathcal{N}$, we have $\mathcal{N}(s) = \pm \mathcal{R}(s)$. For a planar curve, the vectors $\mathcal{T}(s)$ and $\mathcal{N}(s)$ are on the same plane for all s, thus $\dot{\mathcal{B}} \equiv 0$. Now

$$\dot{\mathcal{B}} = \dot{\mathcal{T}} \wedge \mathcal{N} + \mathcal{T} \wedge \dot{\mathcal{N}} = \kappa \mathcal{N} \wedge \mathcal{N} + \mathcal{T} \wedge \dot{\mathcal{N}} \tag{4.7}$$

or

$$\dot{\mathcal{B}} = \mathcal{T} \wedge \dot{\mathcal{R}} \tag{4.8}$$

Since $\dot{\mathcal{R}}(s)$ lies on the tangent plane $T_{\omega(s)}X$, we have

$$\dot{\mathcal{R}} = \gamma_1 \mathcal{T} + \gamma_2 \mathcal{S} \tag{4.9}$$

and thus

$$\dot{\mathcal{B}} = \gamma_2 \mathcal{R} \tag{4.10}$$

Therefore, $\dot{\mathcal{B}} \equiv 0$ if and only if $\gamma_2 \equiv 0$, or $\dot{\mathcal{R}}(s) = \gamma_1(s)\mathcal{T}(s)$. Therefore $c(s)$ is a planar curve if and only if $c(s)$ is a line of curvature.

□

Combining the results from Chapter 2, we have the following proposition:

Proposition 4.2.2. *Let $X : \Omega \subset \mathbb{C} \to \mathbb{R}^3$ be a nonconstant regular minimal surface and $X^* : \Omega \to \mathbb{R}^3$ be an adjoint minimal surface of X. Both X and X^* has the* *same normal mapping $N : \Omega \to S^2$. Let also that $\omega : I \subset \mathbb{R} \to \Omega$ be a C^3-curve with $\dot{\omega}(t) \neq 0$. Then $c = X \circ \omega$ and $c^* = X^* \circ \omega$ are two C^3-curve in X and X^* respectively. Moreover, c and c^* has the same spherical image $\gamma = N \circ \omega$.*

1. *Let $c(t)$ be contained in some straight line L. Then $c(t)$ is both an geodesic and asymptotic line in X, and thus $\gamma(I)$ is contained in some great circle C in S^2. Moreover, c^* is a planar geodesic of X^*.*

2. *If $c(t)$ is a planar geodesic in X, or it is contained in the orthogonal intersection of X with some plane E, then $\gamma(I)$ lies in a great circle in S^2. The curve c^* is a straight arc, thus geodesic asymptotic, in X^*.*

4.3 Schwarzian Chain Problem

Let $X : \Omega_0 \to \mathbb{R}^3$ be a nonconstant minimal surface, and Ω be a simply connected subdomain such that $\bar{\Omega} \subset \Omega_0$. Suppose also that the Gauss map $N : \Omega \to S^2$ is injective, and the boundary of $X(\Omega)$ consists of finitely many straight lines and planar geodesics, i.e. orthogonal intersections of X with planes. In other words, the minimal surface is spanned into a frame $\mathfrak{C} = \{L_i, E_j\}, i = 1, \dots, m, j = 1, \dots, n$ where L_i's are straight lines and E_j's are planes. The set $\mathfrak{C}$ is called a *Schwarzian Chain* , and we say that the surface X is a minimal surface that solves the Schwarzian chain problem for the chain $\mathfrak{C}$. Note that X is perpendicular to all planar parts of the chain $\mathfrak{C}$. In this section, we will show how to construct the Weierstrass function $\mathcal{F}(w)$ of the surface X given a chain $\mathfrak{C}$.

Now consider the map

$$\Omega \xrightarrow{X(u,v)} X \xrightarrow{N} S^2 \xrightarrow{\sigma} \Omega^* \tag{4.11}$$

Since straight lines and planar curve are geodesics in X, they will be mapped to great circles by the Gauss map $N : \Omega \to S^2$. Now consider the meromorphic function ν in the Weierstrass-Enneper Representation

$$\omega = \nu(w), \omega \in \Omega^*, w \in \Omega. \tag{4.12}$$

The map is biholomorphic since

1. $N(w)$ is not the north pole;

2. N is injective.

As stereographic projection maps circles on S^2 to circles in Ω_0. Therefore, the image of straight lines and planar curves in the Schwarzian chain $\mathfrak{C}$ under the map $\nu : \Omega \to \Omega^*$ consists of circular arcs. Since the map ν is biholomorphic, we define $\tau = \nu^{-1}$ such that $\tau(\omega) = w$. Therefore we can find a function $\mathcal{F}(\omega)$ such that

$$\begin{array}{ll} X(w) & = Y(\tau(\omega)) = \Re \int_{\omega_0}^{\omega} \Phi(\omega)\ d\omega \\ \Phi & = [(1-\omega^2)\mathcal{F}(\omega), i(1+\omega^2)\mathcal{F}(\omega), 2\omega\mathcal{F}(\omega)] \end{array} \tag{4.13}$$

Define the function $l(w) = e(w) - if(w)$ which characterizes asymptotic lines and curvature lines and

$$p(w) = \int_{w_0}^{w} \sqrt{l(w)}\ dw \tag{4.14}$$

for some fixed $w_0 \in \Omega$. Since $p'(w) = \sqrt{l(w)}$, then the map

$$\zeta = p(w) \tag{4.15}$$

is a conformal mapping of Ω onto some domain Ω^{**} in the ζ-plane. Note that

$$d\zeta = p'(w) = \sqrt{l(w)(dw)^2}. \tag{4.16}$$

As asymptotic lines are given by $\Re l(w)(dw)^2 = 0$, and $\Im l(w)(dw)^2 = 0$ characterizes the curvature line, so the ζ images of asymptotic lines intersect the real axis at an angle $\pi/4$ or $3\pi/4$, and the curvature lines are straight lines that are parallel to either the real axis or the imaginary axis.

For the minimal surface $X : \bar{\Omega} \to \mathbb{R}^3$ that solves a Schwarzian chain $\mathfrak{C}$ that consists of asymptotic and curvature lines, the map $\zeta = p(w)$ maps Ω onto some polygonal domain Ω^{**} in ζ-plane.

If we compose $\tau : \Omega^* \to \Omega^{**}$ with the map $p : \Omega \to \Omega^{**}$, then the map $q = p \circ \tau : \Omega^* \to \Omega^{**}$ is conformal. Then

$$\begin{aligned} q(w) = p(\tau(\omega)) \quad &= \int_{\omega_0}^{\omega} \sqrt{l(\tau(\omega))}\tau'(\omega)\ d\omega \\ &= \int_{\omega_0}^{\omega} \sqrt{-\mu(\tau(\omega))\nu'(\tau(\omega))\tau'(\omega)^2}\ d\omega \\ &= \int_{\omega_0}^{\omega} \sqrt{-2\mathcal{F}(\omega)}\ d\omega \end{aligned} \tag{4.17}$$

.

Therefore

$$\mathcal{F}(\omega) = \frac{1}{2}\left(\frac{dq(\omega)}{d\omega}\right)^2 \tag{4.18}$$

From our assumptions, the map $\tau : \Omega^* \to \Omega$ is 1-1. If we choose a branch in Ω such that the map $p : \Omega \to \Omega^{**}$ is 1-1, then $q = p \circ \tau$ is a biholomocphic bijective mapping of Ω^* into Ω^{**}, and the extension of the mapping into $\bar{\Omega}^*$maps the vertices of the circular polygonal domain in Ω^* into the vertices of the polygonal domain Ω^{**}. Therefore we have found a method to solve the Schwarzian chain problem by explicit formulas.

Chapter 5

Exploring Minimal Surfaces Using Computer Programs

Computer programs provides visualization of minimal surfaces, and thus are useful in understanding some of the concepts in the minimal surface theory. In this chapter, we will show how to use Maple to draw minimal surfaces using the Weierstrass-Enneper representations. Another program called "Evolver" can be used to solve the Schwarzian chain problem using the area-minimizing principle.

5.1 Maple and Minimal Surfaces

This section is based on [5] which provides an excellent descriptin of how to use Maple to illustrate concepts about minimal surfaces.

According to Eq. (2.1), we can construct minimal surfaces if we have a holomorphic function μ and a meromorphic function ν to generate minimal surfaces. On the other hand, a holomorphic function $\mathcal{F}$ can also be used to serve the same purpose by Eq. (2.3).

We begin by opening a blank Maple worksheet and type in

```
> with(plots):  with(linalg):
```

We will make sure that Maple understands the difference between complex variables z and real variables u and v by entering

```
> assume(u, real); additionally (v,real); additionally (t,real); is(u, real);
is(v, real);
```

$$true$$

$$true$$

The following sub-procedure returns the complex integrals of the Weierstrass-Enneper representation:

```
> Weierz := proc(F)
local Z1, Z2, Z3, Z;
Z1 := int(F(1-z^2), z);
Z2 := int (I * F * (1+z^2), z);
Z3 := int (2  F  z , z);
Z := [Z1, Z2, Z3];
end:
```

Below is a procedure in Maple to generate minimal surfaces using μ and ν. The two functions are called f and g respectively in the procedure. There is a variable called a in the procedure, which is used to simplified the expression by changing the variable z to e^z if we set $a = 1$, and z to $e^{-iz/2}$ when $z = 2$. When a is set to any value other than 1 or 2, then the procedures would not use any substitution for z.

```
Weierfg := proc(f, g, a)
local Z1, Z2, X1, X2, X3, Z3, X;
Z1 := int(f*(1-g^2), z);
Z2 := int(I * f* (1+g^2), z);
Z3 := int(2*f*g, z);
if a=1 then Z1 := subs(z = exp(z), Z1); Z2 := subs (z = exp(z), Z2); Z3
:= subs(z = exp(z), Z3) fi;
if a=2 then Z1 := subs(z = exp(-I*z/2), Z1);
Z2 := subs(z = exp(-I*z/2), Z2); Z3 := subs(z = exp(-I*z/2), Z3) fi;
X1 := simplify(convert(simplify ( Re(evalc(subs(z = u+I*v,
expand(simplify(Z1))))), trig), trig), trig);
```

```
X2 := simplify(convert(simplify ( Re(evalc(subs(z = u+I*v,
expand(simplify(Z2))))), trig), trig), trig);
X3 := simplify(convert(simplify ( Re(evalc(subs(z = u+I*v,
expand(simplify(Z3))))), trig), trig), trig);
X := [X1, X2, X3];
end:
```

Similarly the procedure `Weier` is written for the Weierstrass-Enneper representation II:

```
Weier := proc(F,a)
local Z1,Z2,X1,X2,X3,Z3,X;
Z1 := int(F*(1-z^2),z);
Z2 := int(I*F*(1+z^2), z);
Z3 := int(2*F*z,z);
if a=1 then Z1 := subs(z = exp(z), Z1); Z2 := subs (z = exp(z), Z2); Z3
:= subs(z = exp(z), Z3) fi;
if a=2 then Z1 := subs(z = exp(-I*z/2), Z1); Z2 := subs(z = exp(-I*z/2),
Z2); Z3 := subs(z = exp(-I*z/2), Z3) fi;
X1 := simplify(convert(simplify ( Re(evalc(subs(z = u+I*v,
expand(simplify(Z1))))), trig), trig), trig);
X2 := simplify(convert(simplify ( Re(evalc(subs(z = u+I*v,
expand(simplify(Z2))))), trig), trig), trig);
X3 := simplify(convert(simplify ( Re(evalc(subs(z = u+I*v,
expand(simplify(Z3))))), trig), trig), trig);
X := [X1, X2, X3];
end:
```

With the two procedures `Weierfg` and `Weier`, we can plot minimal surfaces easily by choosing appropriate functions. For example, since we know that $\mathcal{F} = \frac{1}{2z^2}$ for catenoid and $\mathcal{F} = \frac{i}{2z^2}$ for helicoid, we can plot these two surfaces by typing the following in the Maple prompt:

```
> Weier(1/2*z^2), 1);
```

$$[-\cos(v)\cosh(u), -\sin(v)\cosh(u), u]$$

```
> plot3d(Weier(1/(2*z^2), 1), u=-1..1, v=0..2*Pi);
```

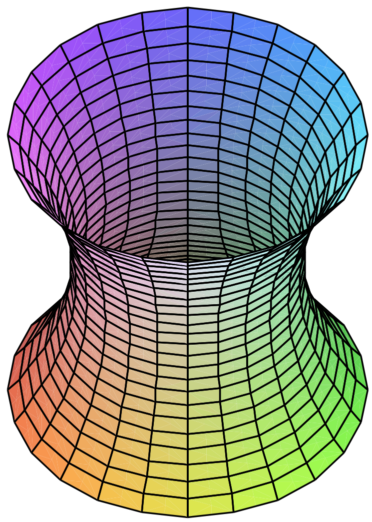


Figure 5-1: $\mathcal{F}(z) = \frac{1}{2z^2}$

For the helicoid, we need a little more steps. The maple function `simplify` has trouble in simplifying $\arctan(\sin(v), \cos(v))$ to v. So when we use the procedure for the helicoid's Weierstrass function $\mathcal{F}(z) = \frac{i}{2z^2}$, one will get

```
> Weier(I/(2*z^2), 1);
```

$$[\sin(v)\sinh(u), -\cos(v)\sinh(u), -\arctan(\sin(v), \cos(v))]$$

Thus when we plot the helicoid, we have to manually change the expression $\arctan(\sin(v), \cos(v))$ to v, and then use the function `plot3d` to plot the result:

```
> plot3d([sin(v)*sinh(u), -cos(v)*sinh(u), -v], u=-1..1, v=0..2*Pi);
```

To obtain the graphs for the associate family of the catenoid, we can use `exp(I*t) * 1/(2*z^2),1;` for the Weierstrass function, and then choose a value for t when

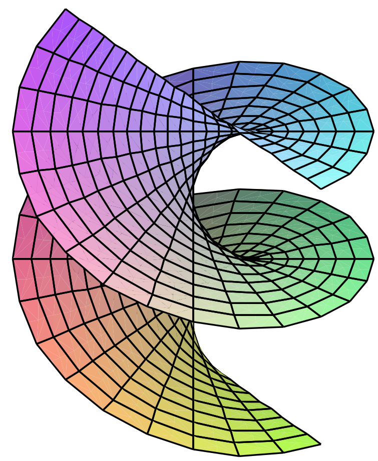


Figure 5-2: $\mathcal{F}(z) = \frac{i}{2z^2}$

plotting the function. For example,

```
> Weier(exp(I*t)*1/(2*z^2),1);
```

$$[(-\cos(t)*\cos(v)*\cosh(u)^2 - \cos(t)*\cos(v)*\cosh(u)*\sinh(u)$$
$$+\sin(t)*\sin(v)*\cosh(u)^2 + \sin(t)*\sin(v)*\cosh(u)*\sinh(u)$$
$$-\sin(t)*\sin(v))/(\cosh(u)+\sinh(u)), -(\sin(t)*\cos(v)*\cosh(u)^2$$
$$+\sin(t)*\cos(v)*\cosh(u)*\sinh(u) - \sin(t)*\cos(v)$$
$$+\cos(t)*\sin(v)*\cosh(u)^2 + \cos(t)*\sin(v)*\cosh(u)*\sinh(u))/(\cosh(u)+\sinh(u)),$$
$$\cos(t)*u - \sin(t)*arctan(\sin(v), \cos(v))]$$

```
> plot3d(subs(t=Pi/4, [(-cos(t)*cos(v)*cosh(u)^2 -cos(t)*cos(v)*cosh(u)*sinh(u)
+sin(t)*sin(v)*cosh(u)^2 +sin(t)*sin(v)*cosh(u)*sinh(u)
-sin(t)*sin(v))/(cosh(u) +sinh(u)), -(sin(t)*cos(v)*cosh(u)^2
+sin(t)*cos(v)*cosh(u)*sinh(u) -sin(t)*cos(v)
+cos(t)*sin(v)*cosh(u)^2 +cos(t)*sin(v)*cosh(u)*sinh(u))/(cosh(u)
```

```
+sinh(u)), cos(t)*u -sin(t)*v]), u=-1..1, v=0..2*Pi);
```

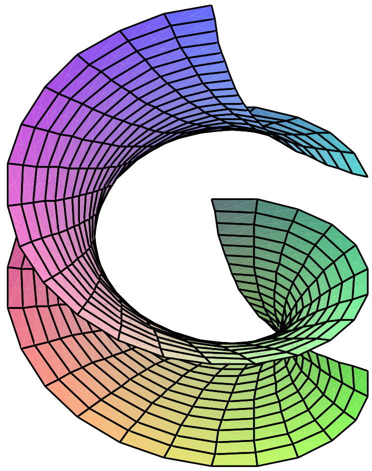


Figure 5-3: $\mathcal{F}(z) = e^{i\pi/4}\frac{1}{2z^2}$

Maple also has a feature that can display sequence of figures, such that we can visualize the bending process of the catenoid into helicoid by displaying a sequence of graphs with the above Weierstrass function. To do that, we substitute the parameter t by $\frac{pi}{2}\frac{i}{30}$, and plot the sequence of figures from $i = 0$ to $i = 30$. That is equivalent of changing t from 0 to $\pi/2$, i.e. changing the catenoid to its adjoint surface. The sub-figures in Figure 2-2 are included in this sequence.

```
> display([seq(plot3d([ -cos(Pi/2*i/30)*cos(v)*cosh(u)
+ sin(Pi/2*i/30)*sin(v)*sinh(u),
-sin(v)*cos(Pi/2*i/30)*cosh(u)-cos(v)*sin(Pi/2*i/30)*sinh(u),
cos(Pi/2*i/30)*u-sin(Pi/2*i/30)*v], u=-1..1, v=0..2*Pi), i=0..30)],
insequence=true);
```

Besides catenoid and helicoid, we can now create other minimal surfaces by the two Maple procedures. For example, the Weierstrass function $\mathcal{F}(z) = \frac{i}{z^3}$ produces the following surface:

```
Weier(I/z^3,2);
```

$$[1/2 * \sin(u)\cosh(v) - 1/2\sin(u)\sinh(v) + \arctan(-\sin(u/2),$$
$$\cos(u/2)), 1/2\cos(u)\cosh(v) - 1/2\cos(u)\sinh(v) - v/2,$$
$$2(\cosh(v/2) - \sinh(v/2))\sin(u/2)]$$

Since the expression contains $\arctan(-\sin(u/2), \cos(u/2))$, we change that to $-u/2$, and plot the surface using the function `plot3d`:

```
> plot3d( [-u/2+1/2*sin(u)*cosh(v)-1/2*sin(u)*sinh(v),
-1/2*v+1/2*cos(u)*cosh(v)-1/2*cos(u)*sinh(v),
2*(cosh(1/2*v)-sinh(1/2*v))*sin(1/2*u)] ,
u=0..6*Pi, v=-1..1, grid=[80,15],
scaling=constrained, orientation=[50,79],shading=XYZ, style=patch);
```

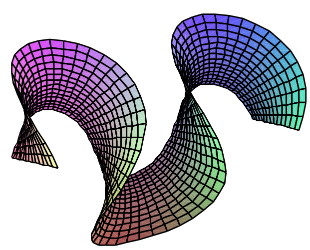


Figure 5-4: $\mathcal{F}(z) = \frac{i}{z^3}$

We can also plot the sequence of its associate surfaces. First we find out the representation of the associate surfaces by

```
> Weier(exp(I*t)*I/z^3,2);
```

$$[1/2*sin(t)*cos(u)*cosh(v) - 1/2*sin(t)*cos(u)*sinh(v)$$
$$+1/2*cos(t)*sin(u)*cosh(v) - 1/2*cos(t)*sin(u)*sinh(v)$$
$$+1/2*sin(t)*v + cos(t)*arctan(-sin(1/2*u), cos(1/2*u)),$$
$$1/2*cos(t)*cos(u)*cosh(v) - 1/2*cos(t)*cos(u)*sinh(v)$$
$$-1/2*sin(t)*sin(u)*cosh(v) + 1/2*sin(t)*sin(u)*sinh(v)$$
$$-1/2*cos(t)*v + sin(t)*arctan(-sin(1/2*u), cos(1/2*u)),$$
$$2*(cosh(1/2*v) - sinh(1/2*v))*(sin(t)*cos(1/2*u)$$
$$+cos(t)*sin(1/2*u))]$$

After fixing the arctan and substituting $t = \frac{\pi}{2}\frac{i}{30}$, we can enter

```
> display([seq(plot3d( [1/2*sin(1/60*Pi*i)*cos(u)*cosh(v)
-1/2*sin(1/60*Pi*i)*cos(u)*sinh(v)+
1/2*cos(1/60*Pi*i)*sin(u)*cosh(v)-1/2*cos(1/60*Pi*i)*sin(u)*sinh(v)+
1/2*sin(1/60*Pi*i)*v-cos(1/60*Pi*i)*u/2,
1/2*cos(1/60*Pi*i)*cos(u)*cosh(v)-1/2*cos(1/60*Pi*i)*cos(u)*sinh(v)-
1/2*sin(1/60*Pi*i)*sin(u)*cosh(v)+1/2*sin(1/60*Pi*i)*sin(u)*sinh(v)
-1/2*cos(1/60*Pi*i)*v-sin(1/60*Pi*i)*u/2,
2*(cosh(1/2*v)-sinh(1/2*v))*(sin(1/60*Pi*i)*cos(1/2*u)
+cos(1/60*Pi*i)*sin(1/2*u))],
u=-0..6*Pi, v=-1..1), i=0..30)], insequence=true);
```

Figures 5-5 to 5-17 are generated using the same method.

5.2 Maple and Björling's Problem

Based on the solution to Björling's problem in Eq. (3.14), we can use Maple to find the minimal surface that solves Björling's problem if the curve $c(t)$ is lying in the xz-plane. The Maple procedure is as below:

```
> with(plots):with(linalg):
> assume(u,real);additionally(v,real); additionally(t,real); is(u,real);
is(v,real);
```

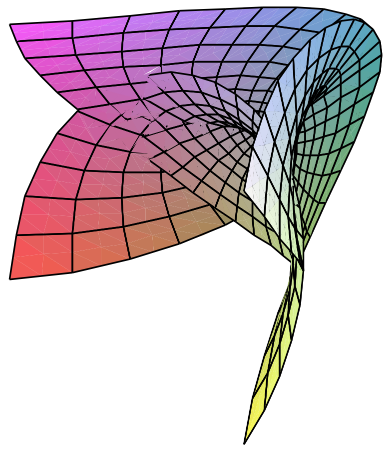


Figure 5-5: $\mathcal{F}(z) = 2z$

$$true$$

$$true$$

```
> Bjor := proc(alpha, a)
local Z1, Z2, X1, X2, X3, Z3, X;
Z1 := subs(t=z, alpha[1]);
Z2 := int(sqrt(diff(subs(t=z, alpha[1]), z)^2 + diff(subs(t=z,alpha[3]),z)^2),z);
Z3 := subs(t=z, alpha[3]);
if a=1 then
Z1 := subs(z=exp(z), Z1);
Z2 := subs(z=exp(z), Z2);
Z3 := subs(z=exp(z), Z3) fi;
if a=2 then
Z1 := subs(z=exp(-I*z/2), Z1);
Z2 := subs(z=exp(-I*z/2), Z2);
Z3 := subs(z=exp(-I*z/2), Z3) fi;
```

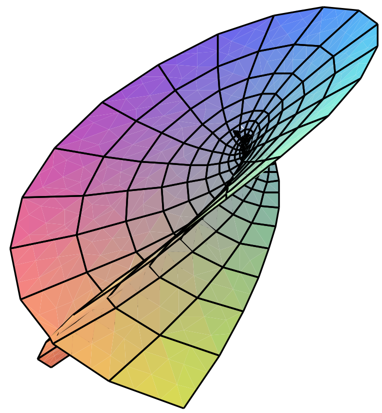


Figure 5-6: $\mathcal{F}(z) = \ln z$

```
X1 := simplify(convert(simplify(Re(evalc(subs(z=u+I*v, expand(simplify(Z1))))),
trig), trig), trig);
X2 := simplify(convert(simplify(Im(evalc(subs(z=u+I*v, expand(simplify(Z2))))),
trig), trig), trig);
X3 := simplify(convert(simplify(Re(evalc(subs(z=u+I*v, expand(simplify(Z3))))),
trig), trig), trig);
X := [X1, X2, X3];
end:
```

Some examples are already shown in chapter 3, such as Catalan surface and Henneberg surface. Here we will give more examples.

The following procedure solves Björling's problem for a parabola parametrized as $(t, 0, t^2)$.

```
> para := Bjor([t, 0, t^2], 1):
> para1:= plot3d(para, u=-1..1, v=0..2*Pi, grid=[15, 35]):
> para2:= spacecurve([t,0,t^2], t=-3..3, color=black, thickness=4):
```

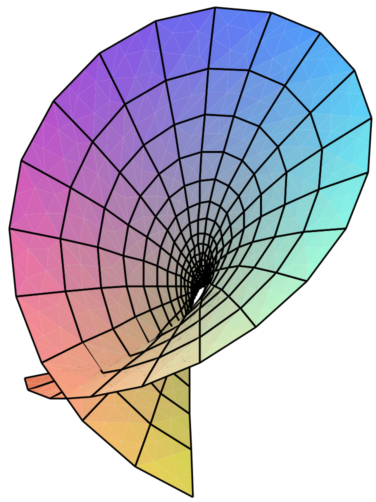


Figure 5-7: $\mathcal{F}(z) = \frac{1}{z}$

```
> display({para1, para2}):
```

Similarly, we can solve Björling's problem for other curves using similar procedures.

5.3 Surface Evolver

Surface Evolver is a powerful program written by K. Brakke. One of the function of the program is to minimize the area of a surface with a given boundary. Since an area-minimizer is a minimal surface, this program provides a way to generate minimal surface in a convenient way. Particularly it is useful to to solve the Schwarzian chain problem, since the boundary of such a problem is easy to be entered to the program. Surface Evolver can also minimize other specified energies over the surface, and it employs a gradient descend method. More information can be found in [2] and [1].

In this section, we will contend ourselves by showing some surfaces that are generated using the Surface Evolver program.

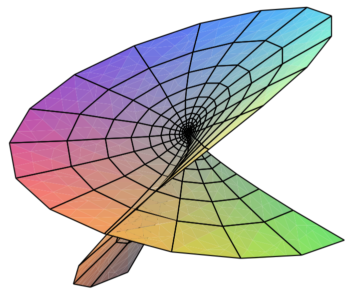


Figure 5-8: $\mathcal{F}(z) = z^2$

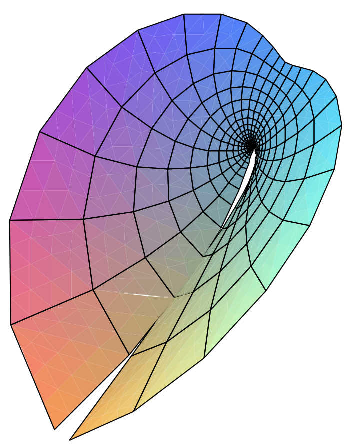


Figure 5-9: $\mathcal{F}(z) = \sin z$

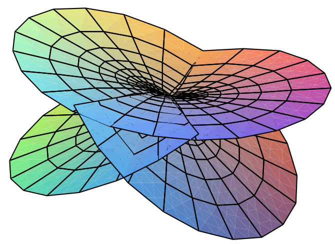


Figure 5-10: $\mathcal{F}(z) = z^4$

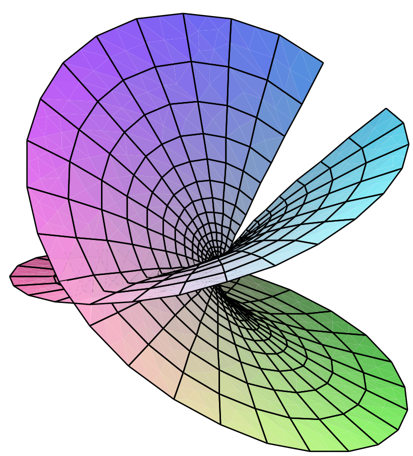


Figure 5-11: $\mathcal{F}(z) = \frac{1}{z^4}$

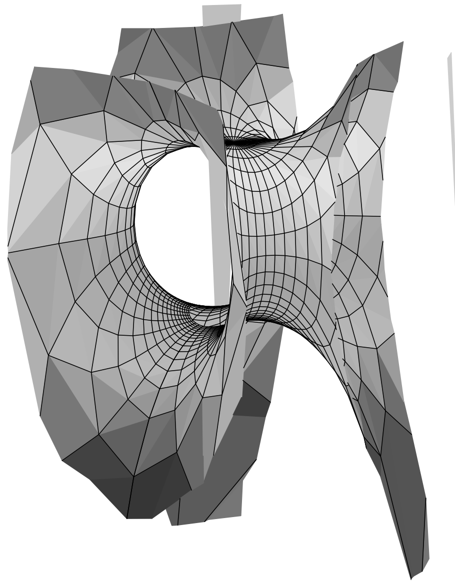


Figure 5-12: Trinoid: $f(z) - \frac{1}{(z^3-1)^2}, g(z) = z^2$

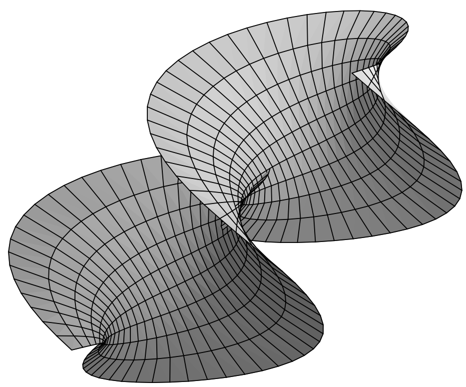


Figure 5-13: $\mathcal{F}(z) = i(\frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3})$

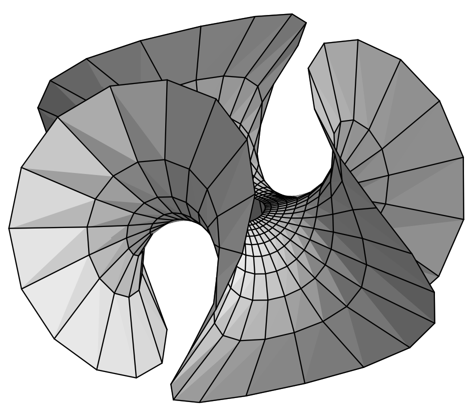


Figure 5-14: $f(z) = \frac{1}{z^2}, g(z) = \frac{1}{z^2}$

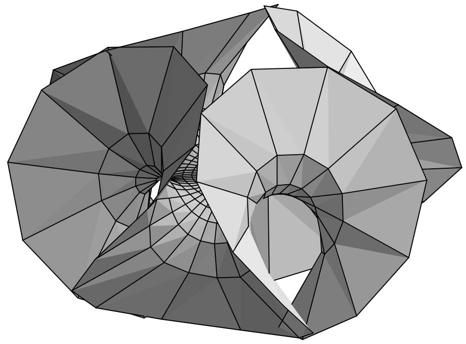


Figure 5-15: $f(z) = \frac{1}{z^2}, g(z) = \frac{1}{z^3}$

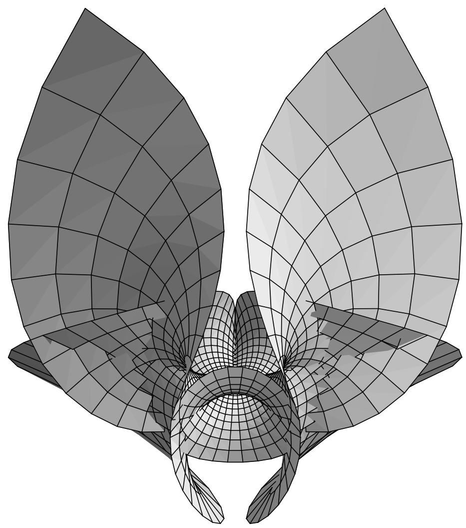


Figure 5-16: $f(z) = z, g(z) = z^3$

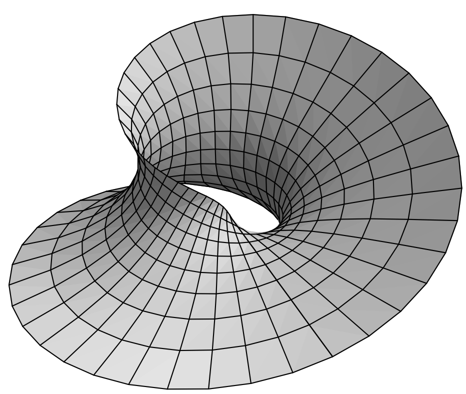


Figure 5-17: $f(z) = \frac{i(z+1)^2}{z^4}, g(z) = \frac{z^2(z-1)}{z+1}$

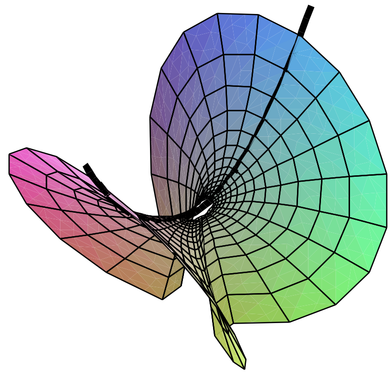


Figure 5-18: A minimal surface solving the Björling's problem for a parabola $z = x^2$.

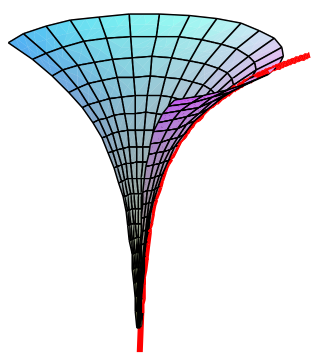


Figure 5-19: A minimal surface solving the Björling's problem for a $z = \ln(x)$.

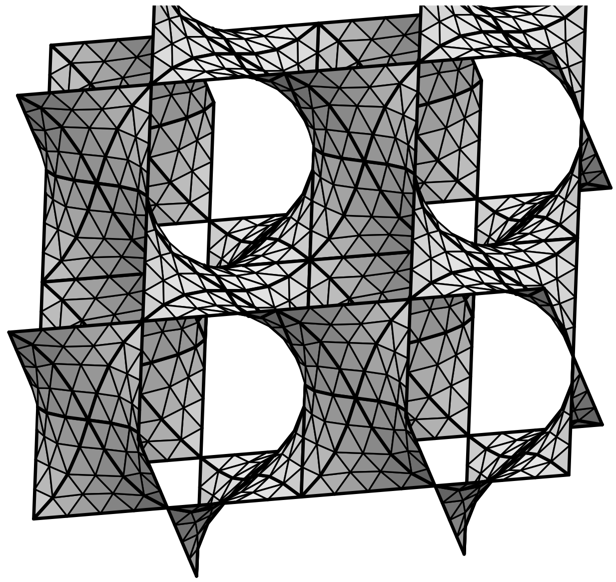


Figure 5-20: Schwarz' D Surface

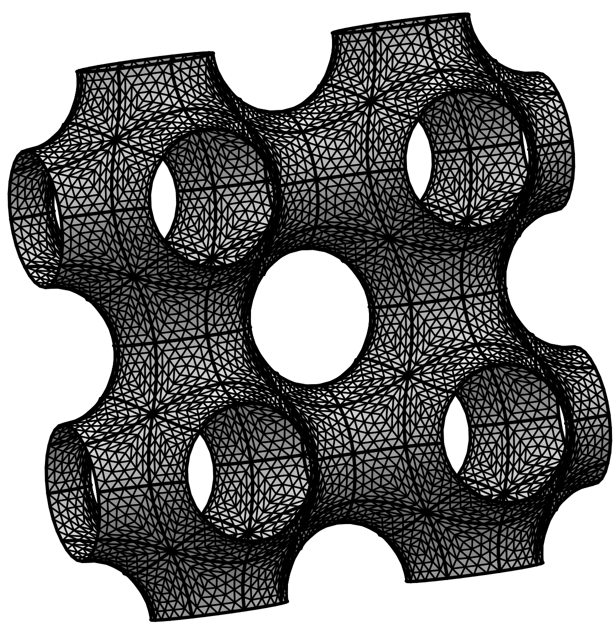


Figure 5-21: Schwarz' P Surface

Appendix A

Some Basic Differential Geometry

In this appendix we will review some of the basic theory in differential geometry about asymptotic lines, curvature lines and geodesics. Readers are referred to [3] for more detailed study.

A.1 Geodesic Curvature and Normal Curvature

Let $\omega : I \subset \mathbb{R} \to \Omega \subset \mathbb{C}$ be a regular curve parametrized by arc length s, and $X(u, v)$ be a regular surface where $w = u + iv \in \Omega$. Then $c(s) = X(\omega(s))$ is a regular curve in the surface X. Denote $\mathcal{T}(s)$ to be the tangent vector $\dot{\omega}(s)$, $\mathcal{R}(s)$ to be the surface normal $N(\omega(s))$, and $\mathcal{S}(s) = \mathcal{R}(s) \wedge \mathcal{T}(s)$ be the *side normal* along the curve ω, then we have a moving orthogonal frame

$$\{\mathcal{T}(s), \mathcal{S}(s), \mathcal{R}(s)\} \tag{A.1}$$

along the curve $\omega(s)$.

We can also define another frame $\{\mathcal{T}(s), \mathcal{N}(s), \mathcal{B}(s)\}$ where $\dot{\mathcal{T}}(s) = \kappa(s)\mathcal{N}(s), \kappa(s) = \|\dot{\mathcal{T}}(s)\|$ and $\mathcal{B}(s) = \mathcal{T}(s) \wedge \mathcal{N}(s)$. The vector $\mathcal{N}(s)$ is called the *principal normal vector*, and $\mathcal{B}(s)$ is called the *binormal vector*.

Since the curve c is parameterized by arc length, we have $\langle \mathcal{T}, \mathcal{T} \rangle = 0$ and thus $\left\langle \mathcal{T}, \dot{\mathcal{T}} \right\rangle = 0$. Since $\{\mathcal{T}(s), \mathcal{S}(s), \mathcal{R}(s)\}$ form an orthogonal basis, we know that $\dot{\mathcal{T}}(s)$

can be written as a linear combination of $\mathcal{S}$ and $\mathcal{R}$:

$$\dot{\mathcal{T}} = \kappa_g \mathcal{S} + \kappa_n \mathcal{R} \tag{A.2}$$

where $kappa_g$ is the *geodesic curvature* and κ_n is the *normal curvature*.

The definition of κ_n in Eq. (A.2) coincides with the definition given in [3][1]. Moreover, $c(s)$ is said to be a *asymptotic line* if $\kappa(s) = 0$ for all $s \in I$.

The value κ_g in Eq. (A.2) can be used to define geodesics.

Definition A.1.1. *The curve $c(s)$ is a* geodesic *if the covariant derivative of $c'(s)$, $Dc'(s)/ds$ vanishes for all $s \in I$.*

The covariant derivative of $c'(s)$ can be defined as the projection of dc/dt onto the tangent plane, and $c'(s) = \mathcal{T}(s)$. Let θ be the angle between $\mathcal{N}$ and $\mathcal{R}$, then

$$cos\theta = \langle \mathcal{N}, \mathcal{R} \rangle \, .$$

Since $\dot{\mathcal{T}}$ lies on the plane spanned by $\mathcal{R}$ and $\mathcal{S}$ and $\mathcal{R}$ is perpendicular to the tangent plane, we have

$$\frac{D\mathcal{T}(s)}{ds} = 0 \Leftrightarrow \left\langle \dot{\mathcal{T}}, \mathcal{S} \right\rangle = 0 \Leftrightarrow \kappa_g = 0 \tag{A.3}$$

[1]See page 141 in [3]

Bibliography

[1] K. Brakke. The surface evolver.

[2] K. A. Brakke. *Surface Evolver Manual*. Mathematics Department, Susquehanna Univerisity, Selinsgrove, PA 17870, 2.24 edition, Oct 2004.

[3] M. Do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice Hall, 1976.

[4] U. Dierkes et. al. *Minimal Surfaces I*, volume 295 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 1992.

[5] John Oprea. *The Mathematics of Soap Films: Explorations with Maple®*, volume 10 of *Student Mathematical Library*. American Mathematical Society, 2000.

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