4.7.2 k-lifts of graphs

Given a graph G, on n nodes and with max-degree Δ, and an integer $k \geq 2$ a random k lift $G^{\otimes k}$ of G is a graph on kn nodes obtained by replacing each edge of G by a random $k \times k$ bipartite matching. More precisely, the adjacency matrix $A^{\otimes k}$ of $G^{\otimes k}$ is a $nk \times nk$ matrix with $k \times k$ blocks given by

$$A_{ij}^{\otimes k} = A_{ij}\Pi_{ij},$$

where Π_{ij} is uniformly randomly drawn from the set of permutations on k elements, and all the edges are independent, except for the fact that $\Pi_{ij} = \Pi_{ji}$. In other words,

$$A^{\otimes k} = \sum_{i<j} A_{ij} \left(e_i e_j^T \otimes \Pi_{ij} + e_j e_i^T \otimes \Pi_{ij}^T \right),$$

where $\otimes$ corresponds to the Kronecker product. Note that

$$\mathbb{E}A^{\otimes k} = A \otimes \left(\frac{1}{k} J \right),$$

where $J = \mathbf{1}\mathbf{1}^T$ is the all-ones matrix.

Open Problem 4.5 (Random k-lifts of graphs) *Give a tight upperbound to*

$$\mathbb{E} \left\| A^{\otimes k} - \mathbb{E}A^{\otimes k} \right\|.$$

Oliveira [Oli10] gives a bound that is essentially of the form $\sqrt{\Delta \log(nk)}$, while the results in [ABG12] suggest that one may expect more concentration for large k. It is worth noting that the case of $k = 2$ can essentially be reduced to a problem where the entries of the random matrix are independent and the results in [BvH15] can be applied to, in some case, remove the logarithmic factor.

References

[ABG12] L. Addario-Berry and S. Griffiths. The spectrum of random lifts. *available at arXiv:1012.4097 [math.CO]*, 2012.

[BvH15] A. S. Bandeira and R. v. Handel. Sharp nonasymptotic bounds on the norm of random matrices with independent entries. *Annals of Probability, to appear*, 2015.

[Oli10] R. I. Oliveira. The spectrum of random k-lifts of large graphs (with possibly large k). *Journal of Combinatorics*, 2010.