# 18.S34 PROBLEMS #7

Fall 2007

74. [1] (a) What is the least number of weights necessary to weigh any integral number of pounds from 1 lb. to 63 lb. inclusive, if the weights must be placed on only one of the scale-pans of a balance? Generalize to any number of pounds.

(b) Same as (a), but from 1 lb. to 40 lb. if the weights can be placed in either of the scale-pans. Generalize.

(c) A gold chain contains 23 links. What is the least number of links which need to be cut so a jeweler can sell any number of links from 1 to 23, inclusive? Generalize.

75. [2.5] A *perfect partition* of the positive integer n is a finite sequence $a_1 \geq a_2 \geq \cdots \geq a_k$ of positive integers a_i, such that each integer $1 \leq m \leq n$ can be written *uniquely* (regarding equal a_i's as indistinguishable) as a sum of a_i's. For instance, there are three perfect partitions of 5, viz., 11111, 221, and 311, since we have the unique representations 1, $1+1$, $1+1+1$, $1+1+1+1$, $1+1+1+1+1$ in the first case; 1, 2, $2+1$, $2+2$, $2+2+1$ in the second; and 1, $1+1$, 3, $3+1$, $3+1+1$ in the third. Show that the number of perfect partitions of n is equal to the number of *ordered factorizations* of $n+1$ into parts greater than one. For instance, the ordered factorizations of 12 are 12, $6 \cdot 2$, $2 \cdot 6$, $4 \cdot 3$, $3 \cdot 4$, $2 \cdot 2 \cdot 3$, $2 \cdot 3 \cdot 2$, and $3 \cdot 2 \cdot 2$, so there are eight perfect partitions of 11.

76. [1] Here is a proof by induction that all people have the same height. We prove that for any positive integer n, any group of n people all have the same height. This is clearly true for $n = 1$. Now assume it for n, and suppose we have a group of $n + 1$ persons, say $P_1, P_2, \ldots, P_{n+1}$. By the induction hypothesis, the n people $P_1, P_2, \ldots, P_n$ all have the same height. Similarly the n people $P_2, P_3, \ldots, P_{n+1}$ all have the same height. Both groups of people contain $P_2, P_3, \ldots, P_n$, so P_1 and P_{n+1} have the same height as $P_2, P_3, \ldots, P_n$. Thus all of $P_1, P_2, \ldots, P_{n+1}$ have the same height. Hence by induction, for any n any group of n people have the same height. Letting n be the total number of people in the world, we conclude that all people have the same height. Is there a flaw in this argument?

77. [1] The following figure consists of three equal squares lined up together, with three diagonals as shown.
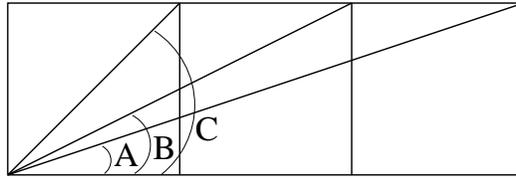
Show that angle C is the sum of angles A and B.

78. [1] Let n be a positive integer.

  (a) Show that if $2^n - 1$ is prime, then n is prime.

  (b) Show that if $2^n + 1$ is prime, then n is a power of two.

  Hint: The simplest way to show that a number is not prime is to factor it explicitly.

79. [2.5] A point p in the interior of an equilateral triangle T is at a distance of 3, 4, and 5 units from the three vertices of T. What is the length of a side of T?
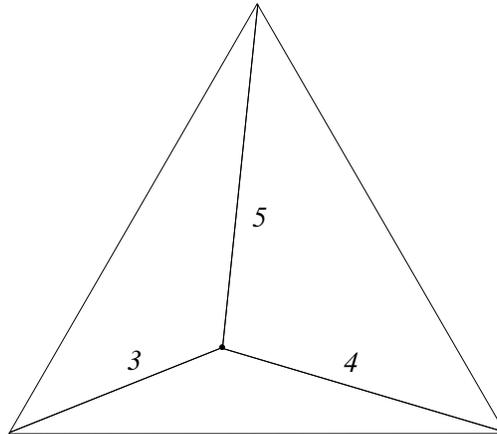
80. [3] Into how few pieces can an equilateral triangle be cut and reassembled to form a square?

81. [2] Let n be an integer greater than one. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not an integer.

82. [2.5]

    (a) Persons X and Y have nonnegative integers painted on their foreheads which only the other can see. They are told that the sum of the two numbers is either 100 or 101. A third person P asks X if he knows the number on his forehead. If X says "no," then P asks Y. If Y says "no," then P asks X again, etc. Assume both X and Y are perfect logicians. Show that eventually one of them will answer "yes." (This may seem paradoxical. For instance, if X and Y both have 50 then Y knows that X will answer "no" to the first question, since from Y's viewpoint X will see either 50 or 51, and in either case cannot deduce his number. So how does either person gain information?)

    (b) Generalize to more than two persons.

83. (2002 Sydney University Mathematical Society Problems competition) [2.5] Let $a(n)$ be the exponent of the largest power of 2 dividing the numerator of $\sum_{i=1}^{n} \frac{2^i}{i}$ (when written as a fraction in lowest terms). For instance, $a(1) = 1$, $a(2) = 2$, $a(3) = 2$, $a(4) = 5$. Show that $\lim_{n\to\infty} a(n) = \infty$.