# 18.S66 PROBLEMS #3a

**Spring 2003**

97. [2] Let $p(n)$ denote the number of partitions of $n$. The number of pairs $(\lambda, \mu)$, where $\lambda \vdash n$, $\mu \vdash n+1$, and the Young diagram of $\mu$ is obtained from that of $\lambda$ by adding one square, is equal to $p(0)+p(1)+\cdots+p(n)$. (Set $p(0)=1$.)

98. [2] Let $e(n)$, $o(n)$, and $k(n)$ denote, respectively, the number of partitions of $n$ with an even number of even parts, with an odd number of even parts, and that are self-conjugate. Then $e(n)-o(n)=k(n)$.

99. [1] The number of partitions of $n$ with $k$ parts equals the number of partitions of $n+\binom{k}{2}$ with $k$ distinct parts.

100. [1.5] The number of partitions of $n \geq 2$ into powers of 2 is even. For instance, when $n=4$ there are the four partitions $4=2+2=2+1+1=1+1+1+1$.

101. [1.5] The *rank* of a partition $\lambda=(\lambda_1,\lambda_2,\ldots)$, denoted $r(\lambda)$, is the size of the main diagonal of the diagram of $\lambda$. Equivalently,

$$r(\lambda)=\#\{i\,:\,\lambda_i \geq i\}.$$

The number of partitions of $n$ of rank $r$ with $r$ parts is equal to the number of partitions of $n$ into $r$ parts which differ by at least 2.

102. [1.5] The number of partitions of $n$ for which no part occurs more than 9 times is equal to the number of partitions of $n$ with no parts divisible by 10.

103. [2] A *perfect partition* of $n \geq 1$ is a partition $\lambda \vdash n$ which "contains" precisely one partition of each positive integer $m \leq n$. In other words, regarding $\lambda$ as the multiset of its parts, for each $m \leq n$ there is a uniqued submultiset of $\lambda$ whose parts sum to $m$. The number of perfect partitions of $n$ is equal to the number of *ordered* factorizations of $n+1$ into integers $\geq 2$.

**Example.** The perfect partitions of 5 are $(1,1,1,1,1)$, $(3,1,1)$, and $(2,2,1)$. The ordered factorizations of 6 are $6 = 2 \cdot 3 = 3 \cdot 2$.

104. [2.5] The number of incongruent triangles with integer sides and perimeter $n$ is equal to the number of partitions of $n-3$ into parts equal to 2, 3, or 4. For example, there are three such triangles with perimeter 9, the side lengths being $(3,3,3), (2,3,4), (1,4,4)$. The corresponding partitions of 6 are 2+2+2=3+3=4+2.

105. [3] The number of partitions of $5n+4$ is divisible by 5.