2.003J/1.053J Dynamics and Control I, Spring 2007
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Lecture 16

Lagrangian Dynamics: Examples

Example: Falling Stick (Continued)

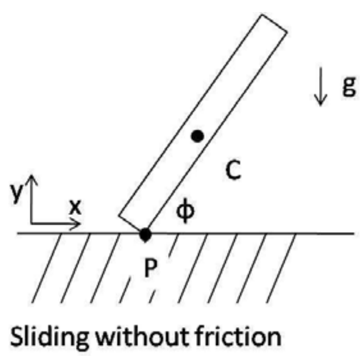

Figure 1: Falling stick. The surface on which the stick rests is frictionless, so the stick slips. Figure by MIT OCW.

A stick slides with out friction as it falls.
Length: L
Mass: M
C: Center of Mass
Assume uniform mass distribution.

x_c, ϕ: Generalized Coordinates

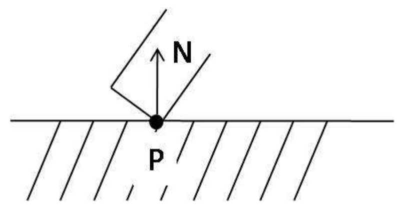

Figure 2: The falling stick is subject to a normal force, N at the point of contact. Figure by MIT OCW.

Holonomic (constraint forces do no work).
No tangential forces at P.
Normal force is a constraint force.
N does no work.

Kinematics

$$\underline{r}_c = x_c\hat{\imath} + y_c\hat{\jmath} = x_c\hat{\imath} + \frac{L}{2}\sin\phi\hat{\jmath}$$

$$\underline{v}_c = \dot{x}_c\hat{\imath} + \frac{L}{2}\dot{\phi}\cos\phi\hat{\jmath}$$

Energy

$$T = \frac{1}{2}m|\underline{v}_c|^2 + \frac{1}{2}I|\underline{\omega}|^2 \quad (1)$$

$$= \frac{1}{2}m(\dot{x}_c^2 + \frac{L^2}{4}\dot{\phi}^2\cos^2\phi) + \frac{1}{2}I_c\dot{\phi}^2 \quad (2)$$

$$(3)$$

$$V = mg\frac{L}{2}\sin\phi$$

Generalized Forces

$\Xi_{x_c} = 0$, $\Xi_\phi = 0$. (Professor Sarma uses Q for generalized forces.) Professor Williams uses Ξ.

Forces: Conservative [gravity] + Nonconservative [normal]. The constraint force (normal force) does no work. As we change x_c and ϕ, no virtual work (no displacement in direction of force).

Lagrangian (L)

$$T - V = L$$

$$L = \frac{1}{2}m(\dot{x}_c^2 + \frac{L^2}{4}\dot{\phi}^2\cos^2\phi) + \frac{1}{2}I_c\dot{\phi}^2 - mg\frac{L}{2}\sin\phi$$

Equations of Motion

We can derive 1 equation per generalized coordinate

For x_c:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_c}\right) - \frac{\partial L}{\partial x_c} = \Xi_{x_c}$$

$\frac{\partial L}{\partial \dot{x}_c} = m\dot{x}_c$, $\frac{\partial L}{\partial x_c} = 0$, $\Xi_{x_c} = 0$.

$$\frac{d}{dt}(m\dot{x}_c) = m\ddot{x}_c \Rightarrow \boxed{m\ddot{x}_c = 0}$$

For ϕ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = \Xi_\phi \tag{4}$$

$$\frac{\partial L}{\partial \dot{\phi}} = \left(\frac{L^2}{2}\dot{\phi}\cos^2\phi\right)\frac{m}{2} + I_c\dot{\phi} \tag{5}$$

$$\frac{\partial L}{\partial \phi} = \frac{1}{2}m\left(-\frac{L^2}{2}\dot{\phi}^2\cos\phi\sin\phi\right) - mg\frac{L}{2}\phi \tag{6}$$

$$\Xi_\phi = 0 \tag{7}$$

Substitute (5), (6), and (7) in (4) to obtain (8):

$$\frac{d}{dt}\left[\left(\frac{mL^2}{4}\cos^2\phi + I_c\right)\dot{\phi}\right] + \frac{mL^2}{4}\dot{\phi}^2\cos\phi\sin\phi + mg\frac{L}{2}\cos\phi = 0 \tag{8}$$

Differentiating the first term results in (10). Substituting in (8) and combining terms yields the equation of motion for ϕ.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left[\left(\frac{mL^2}{4}\cos^2\phi + I_c\right)\dot{\phi}\right] \tag{9}$$

$$= \left(\frac{mL^2}{4}\cos^2\phi + I_c\right)\ddot{\phi} + \left(-\frac{mL^2}{2}\cos\phi\sin\phi\dot{\phi}\right)\dot{\phi} \tag{10}$$

$$\left(\frac{mL^2}{4}\cos^2\phi + I_c\right)\ddot{\phi} - \frac{mL^2}{2}(\cos\phi\sin\phi)\dot{\phi}^2 + \frac{mL^2}{4}(\cos\phi\sin\phi)\dot{\phi}^2 + \frac{mgL}{2}\cos\phi = 0 \tag{11}$$

$$\boxed{\left(I_c + \frac{mL^2}{4}\cos^2\phi\right)\ddot{\phi} - \frac{mL^2}{4}(\cos\phi\sin\phi)\dot{\phi}^2 + \frac{mgL}{2}\cos\phi = 0}$$

This equation is nonlinear.

Equation of Motion by Momentum Principles

Let us derive the equations of motion using momentum principles as a comparison.

Forces

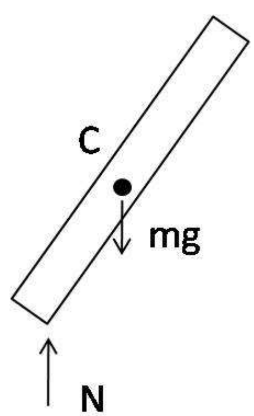

Figure 3: Free body diagram of falling stick. Two forces act on the stick, a normal force, N and a gravitational force, mg. Figure by MIT OCW.

Apply *linear momentum principles*:

$$\sum \underline{F}^{ext} = \frac{d}{dt}\underline{P}$$

Here $\underline{P} = m\dot{x}_c\hat{\imath} + m\frac{L}{2}\dot{\xi}\cos\xi\hat{\jmath}$.

$$\frac{d\underline{P}}{dt} = m\ddot{x}_c\hat{\imath} + m(\frac{L}{2}\ddot{\xi}\cos\xi - \frac{L}{2}\dot{\xi}^2\sin\xi)\hat{\jmath}$$

There are no forces in the x-direction, therefore

$$\boxed{\ddot{x}_c = 0}$$

In the y-direction:

$$N - mg = m\frac{L}{2}\ddot{\xi}\cos\xi - \frac{L}{2}\dot{\xi}^2\sin\xi \qquad (12)$$

We need to eliminate N, so use the *angular momentum principle*.

$$\tau_c^{ext} = \frac{d}{dt}H_c = \frac{d}{dt}(I_c\dot{\xi})$$

$$-N\frac{L}{2}\cos\xi = I_c\ddot{\xi} \qquad (13)$$

After combining equations (12) and (13) and algebra:

$$\boxed{(I_c + \frac{mL^2}{4}\cos^2\xi)\ddot{\xi} - \frac{mL^2}{4}\dot{\xi}^2\sin\xi\cos\xi + mg\frac{L}{2}\cos\xi = 0}$$

Thus, we have derived the same equations of motion. Some comparisons are given in the Table 1.

Advantages of Lagrange	**Disadvantages of Lagrange**
Less Algebra	No consideration of normal forces
Scalar quantities	Less feel for the problem
No accelerations	
No dealing with workless constant forces	

Table 1: Comparison of Newton and Lagrange Methods

Example: Simple Pendulum

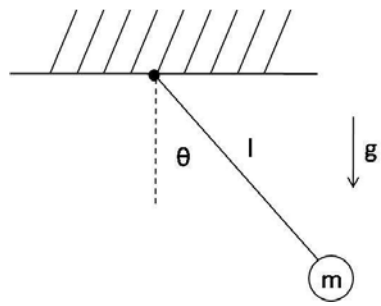

Figure 4: Simple pendulum. The length of the pendulum is l. Figure by MIT OCW.

Kinematics

θ is the generalized coordinate.
Holonomic system (nomral force at P does not move as θ changes. Does no work).

Energy

$$T = \frac{1}{2}m(l\dot{\theta})^2$$
$$V = -mgl\cos\theta$$

Lagrangian

$$L = T - V = \frac{1}{2}m(l\dot{\theta})^2 + mgl\cos\theta$$

Generalized Forces

$$\Xi_\theta = 0 \text{ (No Generalized Forces)}$$

Equations of Motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \Xi_\theta$$

$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$, $\frac{\partial L}{\partial \theta} = -mgL\sin\theta$, $\Xi_\theta = 0$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$

$$\boxed{ml^2\ddot{\theta} + mgL\sin\theta = 0}$$

Alternative View

What if we did not realize that gravity is a conservative force?

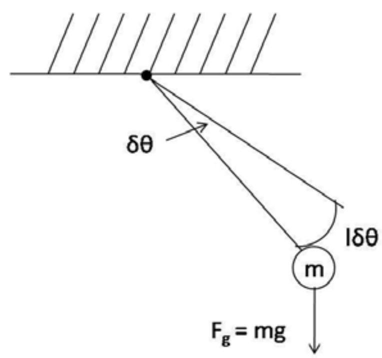

Figure 5: Moving pendulum. When the pendulum rotates by $\delta\theta$, the distance traversed is $l\delta\theta$. Figure by MIT OCW.

What happens to Lagrange's Equations?

Lagrangian

$$T = \frac{1}{2}m(l\dot{\theta})^2$$
$$V = 0$$
$$L = T - V = \frac{1}{2}m(l\dot{\theta})^2$$

No potential forces, because gravity is not conservative for the argument.

Virtual Work and Generalized Forces

$$\delta w^{NC} = -mgl\sin\theta\delta\theta$$

$$\Xi_\theta = -mgl\sin\theta$$

When you displace by $\delta\theta$ the displacement will have a vertical component. It is in the opposite direction of the force.

Equations of Motion

$\frac{\partial L}{\partial\dot\theta} = ml^2\dot\theta$, $\frac{\partial L}{\partial\theta} = 0$,$\Xi_\theta = -mgl\sin\theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = \Xi_\theta$$

$$mL^2\ddot\theta - 0 = -mgl\sin\theta$$

$$\boxed{mL^2\ddot\theta + mgl\sin\theta = 0}$$

Same equation.

It does not matter if you recognize a force as being conservative, just do not account for the same force in both V and Ξ.

Example: Cart with Pendulum and Spring

3 degrees of freedom

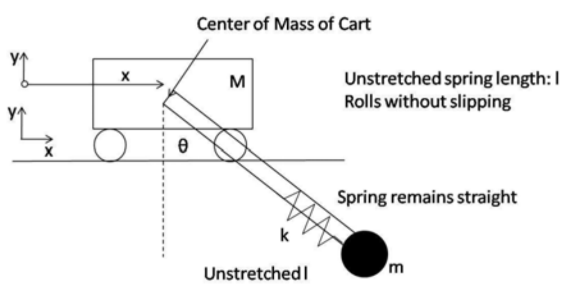

Figure 6: Cart with pendulum and spring. Figure by MIT OCW.

Kinematics

Three generalized coordinates x, θ, s

Cart:

$$x_m = x$$
$$y_m = 0$$

Pendulum:

$$x_m = x + s\sin\theta$$
$$\dot{x}_m = \dot{x} + \dot{s}\sin\theta + s\cos\theta\dot{\theta}$$

$$y_m = -s\cos\theta$$
$$\dot{y}_m = -\dot{s}\cos\theta + s\sin\theta\dot{\theta}$$

Energy

Kinetic Energy (T):

$$\frac{1}{2}M\dot{x}^2 + \frac{1}{2}[(\dot{x} + \dot{s}\sin\theta + s\cos\theta\dot{\theta})^2 + (s\sin\theta\dot{\theta} - \dot{s}\cos\theta)^2]$$

Potential Energy (V):
2 conservative forces - Spring and Gravity

$$V = -mgs\cos\theta + \frac{1}{2}k(s-l)^2$$

Generalized Forces in System

No work from normal forces because cart rolls without slipping.

$$\Xi_x = \Xi_\theta = \Xi_s = 0$$

Lagrangian

$$L = T - V = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m(\dot{s}^2 + s^2\dot{\theta}^2 + 2\dot{x}(\dot{s}\sin\theta + s\dot{\theta}\cos\theta)) + mgs\cos\theta - \frac{1}{2}k(s-l)^2$$

Equations of Motion

For Generalized Coordinate x

x:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \Xi_x$$

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m(\dot{s}\sin\theta + s\dot{\theta}\cos\theta)$$

$\frac{\partial L}{\partial x} = 0$, $\Xi_x = 0$

$$\frac{d}{dt}[(M+m)\dot{x} + m(\dot{s}\sin\theta + s\dot{\theta}\cos\theta)] = 0$$

$\dot{x}(t)$, $\dot{s}(t)$, $s(t)$, $\theta(t)$ are all functions of time.

$$\boxed{(M+m)\ddot{x} + m\ddot{s}\sin\theta + m\dot{s}(\cos\theta)\dot{\theta} + m\dot{s}\dot{\theta}\cos\theta + ms\ddot{\theta}\cos\theta - ms\dot{\theta}(\sin\theta)\dot{\theta} = 0}$$

For Generalized Coordinate θ

θ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \Xi_\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ms^2\dot{\theta} + m\dot{x}s\cos\theta$$

$$\frac{\partial L}{\partial \theta} = m\dot{x}\dot{s}\cos\theta - m\dot{x}s\dot{\theta}\sin\theta - mgs\sin\theta,\ \Xi_\theta = 0$$

$$\frac{d}{dt}[ms^2\dot{\theta} + m\dot{x}s\cos\theta] - m\dot{x}\dot{s}\cos\theta + m\dot{x}s\dot{\theta}\sin\theta + mgs\sin\theta = 0$$

$$2ms\dot{s}\dot{\theta} + ms^2\ddot{\theta} + m\ddot{x}s\cos\theta + m\dot{x}\dot{s}\cos\theta - m\dot{x}s\sin\theta\dot{\theta} - m\dot{x}\dot{s}\cos\theta + m\dot{x}s\dot{\theta}\sin\theta + mgs\sin\theta = 0$$

$$\boxed{s\ddot{\theta} + 2\dot{s}\dot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0}$$

For Generalized Coordinate s

$$\boxed{m\ddot{s} + m\ddot{x}\sin\theta - ms\dot{\theta}^2 - mg\cos\theta + k(s-l) = 0}$$