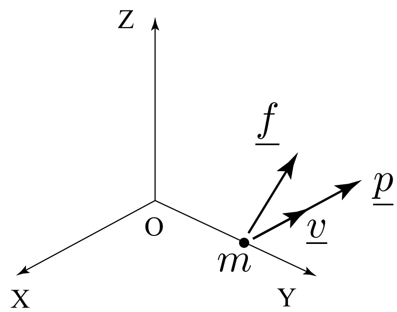
2.003SC
Recitation 4 Notes: Torque and Angular Momentum, Pendulum with Torsional Spring, Rolling Pipe on Moving Truck

Torque and Angular Momentum of a Particle

The figure below shows a fixed coordinate system $OXYZ$ containing a mass m moving with velocity $\underline{v}$, having momentum $\underline{p}$, and being acted upon by a resultant force, $\underline{f}$.



Newton's Law states that the resultant force on the particle equals the time rate of change of the particle's linear momentum.

$$\underline{f} = \frac{d\underline{p}}{dt} \tag{1}$$

where

$$\underline{p} = m\underline{v}$$

Torque and Angular Momentum

If we define a vector, $\underline{r}_o$, from the origin to the particle, and we use it to take the cross product of both sides of (1), equation (1) becomes

$$\underline{r}_o \times \underline{f} = \underline{r}_o \times \frac{d\underline{p}}{dt} \tag{2}$$

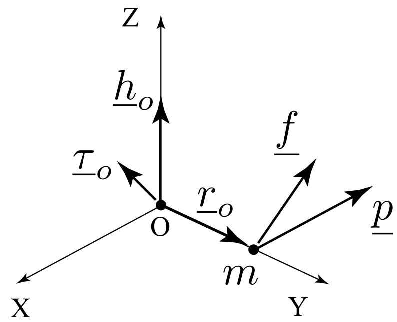
If we define the **torque** and the **angular momentum**, both **about point O** , respectively, as

$$\underline{\tau}_o = \underline{r}_o \times \underline{f} \qquad \underline{h}_o = \underline{r}_o \times \underline{p} \tag{3}$$

Because point O is stationary, equation(2) can be shown to become

$$\underline{\tau}_o = \frac{d\underline{h}_o}{dt} \tag{4}$$

as shown in the figure below.



Note that

- $\underline{\tau}_o$ is perpendicular to the plane formed by $\underline{r}_o$ and $\underline{f}$
- $\underline{h}_o$ is perpendicular to the plane formed by $\underline{r}_o$ and $\underline{p}$

Conservation of Angular Momentum

A direct consequence of equation (4) is that, **if there are no external torques on a system, its angular momentum is conserved.**

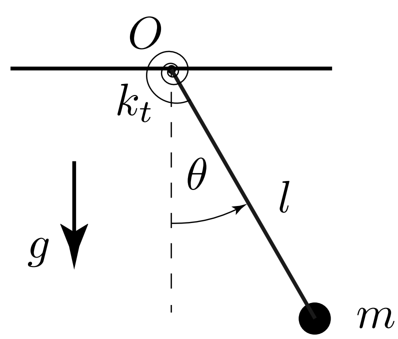
$$\frac{d\underline{h}_o}{dt} = 0 \qquad \rightarrow \qquad \underline{h}_o = \text{constant}$$

Pendulum with Torsional Spring - Problem Statement

A simple pendulum shown below contains a torsional spring at the pivot which creates a restoring torque proportional to θ, i.e. the spring's constitutive relation is $\tau_k = k_t\theta$. Note that θ is positive in the counter-clockwise direction and that the spring constant, k_t , has units of $[N - m/radian]$.

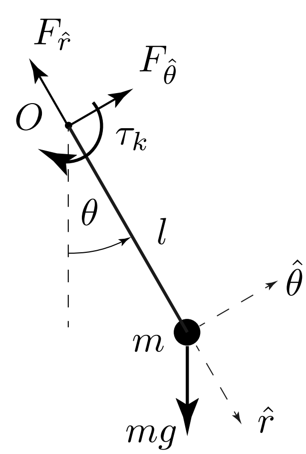
The rod is rigid and without mass. The mass at the end of the rod may be considered a particle of mass, m . Gravity acts.

Find the equation of motion of this pendulum **by taking the time rate of change of the angular momentum computed with respect to the pivot**. Be sure to include a free body diagram.



Pendulum with Torsional Spring - Solution

The free body diagram depicting the torques on the body is shown below. Note the directions of the unit vectors $\hat{r}$ and $\hat{\theta}$. Note also the reaction forces at O.



The position and velocity of the mass are:

$$\underline{r} = l\hat{r} \qquad ; \qquad \underline{v} = l\dot{\theta}\hat{\theta}$$

The angular momentum about point O is given by:

$$\underline{h}_o = \underline{r} \times m\underline{v} = l\hat{r} \times ml\dot{\theta}\hat{\theta} = ml^2\dot{\theta}\hat{k}$$

Summing the external torques about point O (noting that the reaction forces do not contribute...),

$$\Sigma\underline{\tau}_o = (-mglsin\theta - k_t\theta)\hat{k}$$

Taking the time derivative of the angular momentum about point O,

$$\frac{d}{dt}\underline{h}_o = ml^2\ddot{\theta}\hat{k}$$

Since

$$\Sigma\underline{\tau}_o = \frac{d}{dt}\underline{h}_o$$

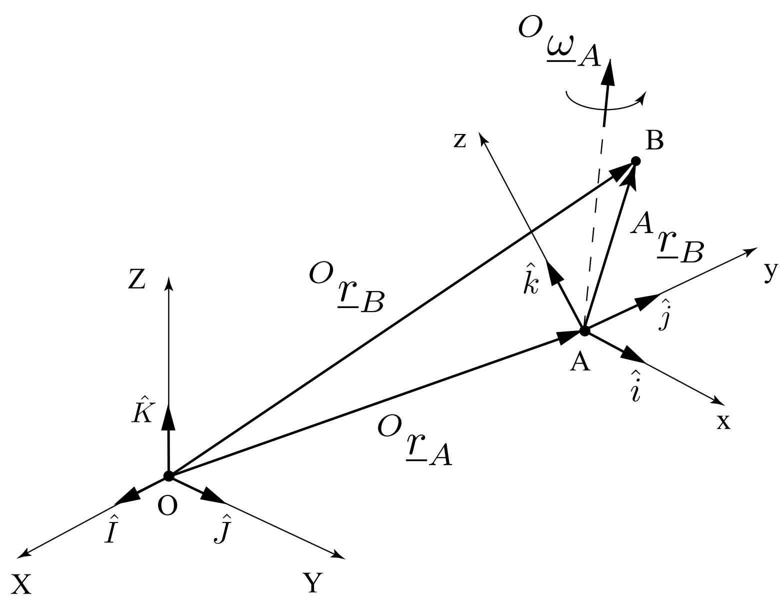
The equation of motion of the pendulum is

$$ml^2\ddot{\theta} + k_t\theta + mglsin\theta = 0$$

Pipe on Bed of Accelerating Truck - Solution

Velocity of a Point in a Moving Frame

Recall the figure and formula below for finding the velocity of a point in a moving frame.



$$^O\underline{V}_B = {}^O\underline{V}_A + {}^A\underline{V}_B|_{^O\underline{\omega}_A=0} + {}^O\underline{\omega}_A \times {}^A\underline{r}_B \tag{1}$$

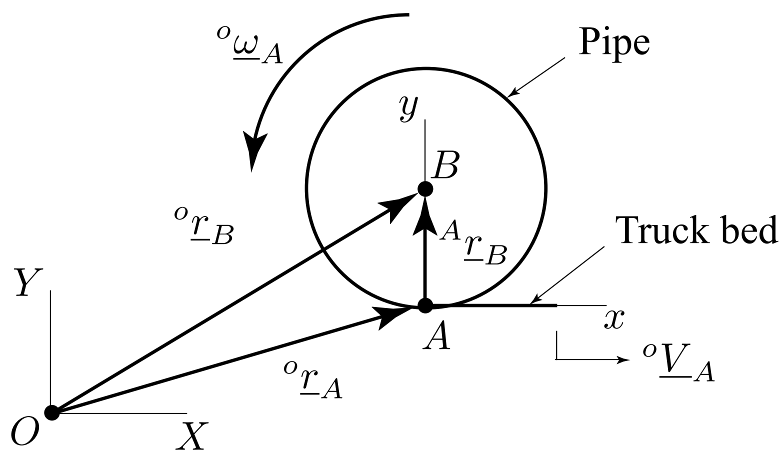
where $Axyz$ is the moving frame which is translating (with velocity $^o\underline{V}_A$) and rotating (with angular velocity $^o\underline{\omega}_A$) with respect to the fixed frame $OXYZ$.

Key Geometry

The relevant geometry from our problem, i.e. the pipe and the truck bed, is shown in the figure below. Note two small but important changes in the problem statement.

- We relabel the (rolling) point of contact between the pipe and the truck bed to be **point** A.
- We relabel the center of the pipe (whose velocity with respect to ground we seek) to be **point** B.

Now choosing $Axyz$ as the moving frame (attached to the pipe) causes our problem to correspond exactly to equation (1), which can be used "as is".



Supplying the values given in the problem statement,

$$^O\underline{V}_A = 3 \text{ m/s } \hat{I}$$
$$^A\underline{V}_B|_{^O\underline{\omega}_A=0} = 0$$
$$^O\underline{\omega}_A = 6 \text{ rad/s } \hat{K}$$
$$^A\underline{r}_B = 1.5 \text{ m } \hat{J}$$

the velocity of point B can be seen to be

$$^O\underline{V}_B = (3 \text{ m/s } \hat{I}) + (0) + (6 \text{ rad/s } \hat{K} \times 1.5 \text{ m } \hat{J})$$

$$^O\underline{V}_B = \text{-6 m/s } \hat{I}$$

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