J. KIM VANDIVER: Today's lecture is not mathematically hard, but it's really important to establish vocabulary today. We're going to talk about vibration for the rest of the term. And vibration is essentially applied dynamics. So up until now, we've been finding equations of motion, but not solving them. Did you notice that? I've almost never asked you to solve the equation of motion that you've just discovered using Lagrange or whatever. The rest of the term, we're actually going to be talking mostly about the resulting motion. The equations of motion are pretty easy to find. You have all the techniques that you need to know for finding. And now, we're going to talk about how things vibrate.

So why do we choose vibration? Vibration, one, is an incredibly common phenomenon. We wouldn't have speech without vibration. You wouldn't have musical instruments without vibration. It's a positive thing when it's making good music. It's a negative thing when it's keeping you awake at night because the air conditioner in the next room is causing something to rattle in the room and it's driving you nuts. So you can want it, it can be desirable, and you cannot want it. And you need to know ways of getting rid of it. And so we're going to talk about vibration, about making vibration, about suppressing vibration, about isolating sensitive instruments from the vibration of the floor, things like that.

So that's the topic of the rest of the term. And today, we're going to talk about single degree of freedom systems. And you might think that we're spending an awful lot of time on single degree of freedom systems. But actually, there's a reason for that. Lots of things in real life, like-- this is just an aluminum rod. This will vibrate. And continuous systems, which this is, have a theoretically infinite number of degrees of freedom. Yet when it comes to talking about its vibration, it is conceptually easy to think about the vibration of an object like this, one natural frequency, one natural mode at a time.

And in fact, you can model that natural mode with its single degree of freedom equivalent. And that's the way I approach vibration. So if you can isolate one particular mode, you can literally model it as a Mass-Spring-Dashpot. So you need to understand the Mass-Spring-Dashpot behavior inside and out, because it's the vocabulary we use to do much more complicated things. So a single degree of freedom system, like the simple pendulum, has a natural frequency. In this case, it has mode shape.

Here's another one, kind of fun, single degree of freedom. This obviously involves rotation. And you can figure that out using Lagrange or whatever, single degree of freedom systems. But now, I'm going to excite one mode of vibration of this.

[CLANG]

[HIGH-PITCHED TONE]

Hear the real high pitch? I'll get it down here by the mic so that people at home can hear it-- about a kilohertz, way up there. And that's one natural mode of this thing in longitudinal vibration. When I thump it sideways--

[CLANG]

[LOWER TONE]

you hear a lower tone. Hear that?

[HUMS LOW]

rather than--

[HUMS HIGH]

[CLANG]

That's bending vibration of this thing. But each mode of vibration I can think of in terms of its equivalent single degree of freedom oscillate. So we'll get to talking about these things a little bit-- continuous systems-- in the last couple lectures of the term. But for today then, we're really going to develop this vocabulary around the vibration of single degree of freedom systems. So let's start.

All right. So to keep it from being totally boring, I'm going to start with a little Mass-Spring-Dashpot that has two springs. And they're of such a length that unstretched, they just meet in the middle. And then, I'm going to take a mass and I'm going to squeeze it in between these two springs-- I can't draw a spring very well today-- and this is k1 and this is k2 and here's m. And we'll put it on a roller so it's obviously constrained to motion in one direction. And I'll pick this point here as the place I'm going to put my inertial coordinate. So my inertial coordinate's just measured from or happens to be where the endpoints of these two springs were.

Now, to squeeze the spring in here, I have this clearly pre-compression in these springs. So we are no longer in a zero-force state, right? So and I want to get the equations of motion in this. And moreover, I want to predict-- I want to find out what's the natural frequency of this spring. So let's check your intuition. So write down on your piece of paper whether or not the natural frequency will be different because there's pre-compression, or whether or not that pre-compression in the springs has nothing to do with the natural frequency. So write down on your paper "natural frequency is different" or "natural frequency is the same." Let's have a prediction here.

And then, we'll set about figuring this out and in the course of doing it, we'll develop a little vocabulary. All through the course so far, when we've done equations of motion, we've usually picked the zero-spring-force position. And we sort of led you down this rosy path that suggests that's the way we do it. But there are other ways that you're going to find that are preferable to that, sometimes. So that's one of the reasons I'm doing this example. So let's do a free body diagram.

And if I held this mass, for example, right at the center when I put the springs in, it's obvious that this spring gets compressed by half of the length of the mass and this spring gets compressed by half of the length of the mass, right? So this is going to be L long. So if I held it right in the middle, it would compress L/2 and L/2.

But then, when I release it, if these springs are a different spring constant, it's going to move a little bit. So the force on this side pushing back is sum k1 times L/2 minus the distance that I move in that direction, which would relieve it. And the force on this side also pushes back. It's k2 times L/2 over 2 plus x, because when I go in that direction, I'm compressing it even further. And those are the total forces in the x-direction on this body. There's an N and an mg, which we know we don't have to deal with because we're only interested in motion left and right. All right?

So we can say sum of the forces in the x-direction, mass times the acceleration. And those forces are k1 L/2 minus x minus k2 L/2 plus x. And that's the complete equation of motion for this problem. And rearrange it so that I get the functions of x together here. So mx double dot plus k1 plus k2 times x equals L/2 times k1 minus k2. And that's your equation of motion. It's non-homogeneous. This is all constants on the right-hand side. And on the left-hand side are the functions of x, right? So what's the natural frequency of the system?

AUDIENCE: Square root of k1 plus k2 over m.

J. KIM VANDIVER: I hear a square root of the quantity k1 plus k2, the stiffness, divided by m, k over m, a usual Mass-Spring-Dashpot system. Did the pre-compression have anything to do with the natural frequency? I won't ask you to embarrass yourselves, but a few of you probably got that wrong, all right?

So there's a lesson in this that I want you to go away with. and I'll say it once. And that is when an external force has nothing to do with the motion coordinates in the problem. It doesn't affect the natural frequency. These come from external forces. These are these pre-compressions, right? And I can separate them out and they are not functions of x. The stuff on the right-hand side of the equation, that's not a function of the motion variable-- cannot affect the natural frequency.

So I'll give you another one. This is our common thing hanging from a stick. I've taken my system I built the other day for a different purpose, but now, it's just a mass hanging from a spring. And it's right now at its equilibrium position or there's non-zero force in the spring. It clearly has a natural frequency. And is that natural frequency a function of gravity? And so if you go to write the equation to motion of this system, you would find mx double dot plus kx equals mg g. But the mg is not a function of x. The natural frequency's again, the square root of k/m.

Now, we want to talk about solving this differential equation. And because it's got this constant term in the right-hand side, it's non-homogeneous, which is kind of a nuisance term in terms of dealing with a differential equation. It'd be a lot nicer if the right-hand side were 0. So I want to make the right-hand side of this one 0. And draw a use of a conclusion from that.

First thing I need to know is I'd like to know what is the static equilibrium position of this. And when you go to compute static equilibrium, you look at the equation of motion and you say, make all motion variables things that are functions of time 0. So no acceleration-- you're left with this. So you just solve this for whatever the value of x is and I'll call it x of s for x-static. And you'll find that, oh, well, it's that term divided by k1 plus k2, k1 minus k2 all over k1 plus k2. And that's the static position.

So now, let's say, ah, well, we started off with this motion variable that wasn't arbitrarily defined at the middle. And let's say that, well, it's made up of a static component, which is a constant, just a value, plus a dynamic component I'll call x of d, which moves. This is the function of time. This is a constant. It's not a function of time.

And that means if I take its derivative, I might need a value for x dot. That goes away. It's just xd dot. And x double dot is xd double dot. And let's substitute this into my equation of motion. So it becomes m xd double dot plus k1 plus k2 times-- and now, this term has got two pieces now-- times xd plus k1 plus k2 times xs equals L/2 k1 minus k2. All right?

Now if I say, well, let's examine the static case, then this goes away. For the static equilibrium case, this term is 0. This term is 0 because the dynamic motion is 0 in the static case. That xd is motion about the static equilibrium position. So for static case, these two terms go away and we know that this equals that. But if that's true, we can get rid of these. They cancel one another. These terms cancel and I'm left with m xd double dot plus k equivalent, I'll call it, xd equals 0. So the k equivalent's just the total stiffnesses in the system, whatever works out, right? In this case, it's k1 plus k2 and the natural frequency, omega n, is the square root of k equivalent divided by m.

So most often, if you're interested in vibration, you're interested in natural frequencies, you're interested in solving the differential equation, you will find it advantageous to write your equations of motion around the static equilibrium position. So I could have started this problem by saying, whatever the static equilibrium position is of this thing, that's what I'm measuring x from. And then, I would have come to this equation eventually. You'd have to figure out what is the static equilibrium position and know what you're doing, but once you know it, then you have the answer.

Now, the same thing is true of that problem. That's a non-homogeneous differential equation for the hanging mass. And we derive the equations of motion things for this many different ways this term, all right? But we usually said, zero-spring force. But now, if you started from here and said, this is the static equilibrium position, what's the motion about this position, then you'd get the equation with 0 on the right-hand side-- lots of advantages there to using that.

All single degree of freedom oscillators will boil down to this equation. This is one involving translation, but for a simple pendulum. This object, for example, is a pendulum, but it's rotational. So it's a pendulum, but it's one degree of freedom. All pendulum problems, if you do them about equilibrium positions, boil down to some I with respect to the point that they're rocking about, theta double dot plus some Kt, torsional spring constant theta, equals 0. They take the same form.

So all translational single degree of freedom systems, all rotational single degree of freedom systems, it's the same differential equation-- just this involves mass and linear acceleration. This involves mass moment of inertia and rotational acceleration. So everything that I say about the solution to single degree of freedom systems applies to both types of problems.

So let's look into the solution of this equation briefly. Mostly, I'm doing this to establish some terminology. So a solution I know or I can show that xd of t, the solution to this problem-- notice, are there any external forces, by the way, excitations, f of t's or anything? No. So this thing has no external excitation that's going to make it move. So it's only source of vibration or motion is what? Comes from-- I hear initial conditions, right? You have to do something to perturb it and then it will vibrate.

So here it is. It's about its equilibrium position. I give it an initial deflection and let go. Or it's around its initial condition and I give it an initial velocity. It also responds to some combination of the two. So initial conditions are the only things that account for motion of something without external excitation. And that motion, I can write that solution as A cosine omega t. You'll find this is a possible solution. B sine omega t is another possible solution. Sum A cosine omega t minus phase angle's also a solution. And sum A e to the i omega t you'll find is also a solution. Any of those things you could throw in and the precise values of these things, the A's, the B's, the phi's, and so forth depend on--

AUDIENCE: the initial conditions.

J. KIM VANDIVER: The initial conditions. So let's do this one quickly. All right. And I'll choose And I'm going to stop writing the x sub d here. This is now my position from the equilibrium point. So x of t-- I'm going to say, let it be an A1 cosine omega t plus a B1 sine omega t and plug it in. When I plug it into the equation of motion, x double dot requires you to take two derivatives of each of these terms. Two derivatives of cosine gives you minus omega cosine. Two derivative sine minus omega squared cosine minus omega squared sine. So the answer comes out minus m omega squared plus k equivalent here times A1 cosine plus B1 sine-- omega t's obviously in them-- equals 0.

So I just plugged in that equation of motion. I get this back. This is what I started with. That's x. In general, it is not equal to 0, can take on all sorts of values. So that's not generally 0 and that means this must be. And from this, then, when we solve this, we find that omega what we call n squared is k over m. And that's, of course, where our natural frequency comes from. This is called the undamped natural frequency, because there's no dampening in this problem yet. We get the square root of k over m is the natural frequency of the system.

Let's find out what are A1 and B1. Well, let's let x0 be x at t equals 0 here. And if we just plug that in here, put t equals 0 here, cosine goes to 1. This term goes away. So this implies that A1 equals x0. So we find out right away that the A1 cosine omega t takes care of the response to an initial deflection. And we need a x dot here minus A1 omega sine omega t plus B1 omega cosine omega t. That's the derivative of x. You know the solution's that, so its first derivative, the velocity, must look like this. And let's let v0 equals x dot at t equals 0.

When we plug that in, this term goes away and we get B1 omega and cosine is 1. So therefore, B1 is v0 over omega. But in fact, the omega's omega n, because we already found that, that the only frequency that satisfies the equation of motion when you have only initial conditions in the system, the only frequency that is allowed in the answer is the natural frequency.

So we now know B1 is v0 over omega n and A1 is x0. So if I give you any combination of initial displacement and initial velocity, you can write out for me the exact time history of the motion. X0 to cosine omega t plus v0 over omega n sine omega t is the complete solution for a response to initial conditions.

So any translational oscillator one degree of freedom where you have a translational coordinate measured from its equilibrium position has the equation of motion-- actually, you've done this enough. But if we added a force here and we added some damping and I wanted the equation of motion of this, you know that it's m x double dot plus b x dot plus kx equals F of t.

And so you're going to be confronted with problems-- find the equation of motion in a system. It comes up looking like that and they say, what's the natural frequency? And I've been a little sloppy. I really mean, what's the undamped natural frequency? And so to find the undamped-- when one says that, what's the undamped natural frequency, you just temporarily let b and F be 0, just temporarily, and solve then for omega n equals square root of k/n. It's what you do. And then, so we know this is a parameter that tells us about the behavior of the system, which we always want to know for the single degree of freedom systems. What is the natural frequency of the system?

And we know for b equals 0 and F of 0, then the response can be only due to initial conditions. So we have x of t. We know it's going to be some x0 cosine omega n t plus v0 over omega n sine omega n t. And every simple vibration system in the world behaves basically like this from initial conditions. It'll be some part responding to the initial displacement, some part to the initial velocity. And damping is going to make it a little bit more complex, but not actually by much. The same basic terms appear even when you have damping in it.

This can be expressed as sum A cosine omega, in this case, n t minus the phase angle. And it's useful to know this trigonometric identity to be able to put things together into an expression like that. And you'll find out that A is just the square root of the two pieces. It's a sine and cosine term. So you have an x0 squared plus a v0 over omega n squared square root.

Remember, this is any A and B. It's just a square root of A squared plus B squared. That's what we're doing here. And the phase angle, the tangent inverse of this-- we've been calling this like an A and this is the B quantity. So tangent inverse of-- get my signs right-- B over A, which in this case then is tangent inverse of v0 over x0 omega n. That's all there is to it.

And finally, another trig thing that you need to know-- we're going to use it quite a bit-- is that if you have an expression A cosine omega t minus phi, that's equal to the real part of A e to the i omega t. And if A is real and-- I don't want to write it that way-- when A is real, it's A times e to the i omega t minus phi, because Euler's formula says e to the i theta equals cosine theta plus i sine of theta. So if you have an i omega t minus phi here, you get back a cosine omega t minus phi and another term, an i sine omega t minus phi. So you can always express that as the real part of that. So we're going to need that little trig identity as we go through the term.

Now, I've found in many years of teaching vibration that something that many students find a little confusing is this notion of phase angle. What does "phase angle" really mean? So I'll try to explain it to you in a couple different ways. So let's look at what this vibration that we're talking about here, x0 cosine omega t plus v0 over omega n sine-- what's it look like? So that's-- we've just got our-- and we see what it looks like. But if you plot the motion of this thing just versus time, what's it look like and where does phase angle come into it?

So this is now x of t and this is t equals 0 and this undamped system is essentially going to look like that. And this is the value x0, the amplitude, the initial condition on x that you began with. And right here, the slope-- v0 is the slope, the initial slope of this curve, right, because the time derivative is F x dot. If we were plotting x dot, the initial velocity is omega x0. And so it's just the slope is v0 here. So this is your initial velocity. This is the-- and I didn't-- yeah, that's right. This is the initial displacement. The total written out mathematically, it looks like that. And I'm plotting this function, A cosine omega t minus phi. Yeah? Did I see a hand up?

AUDIENCE: Does x0 at t equals 0 or is it a little bit after?

J. KIM VANDIVER: Well, I was just looking at it myself and said, this can't be right. This has got to be the initial condition on x and this has to be the initial condition on v. Now, whatever this turns out to be is whatever it turns out to be. You have some initial velocity. You have some initial displacement. The system can actually peak out sometime later at a maximum value, right? And that maximum value is that. So this over here is the square root of x0 squared plus v0 over omega n squared square root. That's what the peak value is. And this system's undamped, so it just goes on forever.

So the question is, though, what is this gap here between when it starts and when it meets its maximum? Well, when we use an expression like-- we said we can express this as some A cosine omega t minus phi. It's just the point at which the cosine then reaches its maximum. So if this axis here is omega t, if we plot this actually versus omega t, then one full cycle here is 2 pi or 360 degrees. So if you plot it versus omega t, then this gap in here is just phi. That's the delay in angle, if you will, that the system goes through between getting from the initial conditions to getting to the peak of the cosine.

And phi must also then be equal to some omega n times a delta tau, I'll call it, some time delay. So if this is plotted-- if this axis is time-- not omega t, but time-- then x the same plot, this delay here, this is a time delay. And when you plot it against time, it's a delay in time to get to the peak. And omega n delta tau, this delay, must be equal to the phase angle. So the delta tau, this time delay, is phi over omega n.

So you can think about this as a delay in time or as a shift in phase angle, depending on whether or not you want to plot this thing as a function of omega t or as a function of time. But you're going to need this concept of phase angle the rest of the term. Want to ask any questions about phase?

Because we're doing vibration for the remainder of the term, this is an introduction to a topic called linear systems. And so this is basically the fundamental stuff in which you then, when you go on to 2004, which is controls and that sort of thing, this is the basic intro to it. And we'll talk more about linear system behavior as we go along.

Now, we're going to do something that you've-- much of this stuff I know you've seen before. Some of the new parts is just vocabulary and ways of thinking about vibration that engineers do that mathematicians tend not to. So you have seen most of this stuff before where?

AUDIENCE: 1.803.

J. KIM VANDIVER: 1.803, right? You've done all this. And a year ago last May, in May, I taught the 1803 lecture with Professor Haynes Miller. Now, if you had 1.803 last spring, I think you had somebody different. But he invited me to come here. It was in the same classroom and we taught the second-order ordinary differential equation together. It was really a lot of fun. He said, well, here's what we do. And then, I said, oh, well, engineers look at it the following way. So what I'm going to show you is what he and I did in class that day. You can go back and watch that on video. It's kind of fun. But I'll give you my take on it today. So this is the engineer's view of what you've already seen in 1.803.

So we have that system and we have that equation of motion. And the engineers and mathematicians would more or less agree to that m x double dot plus bx. But I went and looked at the web page last night. Last spring, the person used c instead of b. Haynes Miller the year before used b. So you can't depend on any absolute consistency. So let's start off with our homogeneous equation here. And I'm looking now for the response to initial conditions with damping. You've done this in 1.803. You know that you can solve this by assuming a solution of a form Ae to the st.

Plugging it in gives you a quadratic equation that looks like s squared plus sb plus k equals 0. This has roots. I left out my m here, so it starts off looking like that. You divide through by the m. s squared plus b/m s plus k/m equals 0. And that's where Haynes would leave it. And he'd give you the entire answer in terms of b/m and k/m and that kind of thing. Engineers, we like to call that the natural frequency squared. And this term, we'd modify to put it in a terminology that is more convenient to engineering. So I'll show you how that works out.

When you solve this quadratic just using the quadratic equation, you get the following. You get that the roots, there's two of them. I'll call them S1 and 2. The roots to this equation look like minus b over 2m plus or minus square root of b squared over 4m squared minus k/m. And that's what you'd get to do in 1.803.

And an engineer would say, well, let's change that a little bit. So my roots that I would use for S1 and 2, I just factor out-- that's omega n squared. I can factor that out and it becomes omega n on the outside. And I put an omega n in the numerator and denominator here, as well. So I get roots that look like-- so I've just manipulated that a little bit. I have a name for this term. I use the Greek letter zeta is b over 2 omega n m is the way I remember it in my brain. It's called the damping ratio. And if I say that, then the roots, S1 and 2 for this, look like minus zeta omega n plus or minus omega n times the square root of zeta squared minus 1.

And those are the roots that a vibration engineer would use to describe this second-order linear differential equation solution homogeneous solution. Those are the roots of the equation. And when you have no damping, then this term goes away and you're left with-- and I left an i out of here, I think. No, I'm fine. The i comes out of here.

So for one thing to absolutely take away from today is to remember this. That's our definition of damping called the damping ratio. When that's 1, it's a number we call critical damping. I'll show you what that means in a second. And when it's greater than 1, the system won't vibrate. It just has exponential decay. If it's less than 1, you get vibration. And that's why we like to use it this way as it's meaningful. Its value, you instantly know if it's greater than or less than 1, it's going to change the behavior of the system from vibrating to not vibrating.

So now, there's four possible solutions to this. I'm not going to elaborate on all of them, but zeta equals 0, we've already done. We know the answer to that. Response to initial conditions-- simple. We know that one. We have another solution when zeta's greater than 1. When zeta's greater than 1, this quantity here is the inside is greater than 1, so it's a real positive number. And both the roots of this thing are completely real.

And you know that the-- remember the response, we hypothesize in the beginning that response looks like some Ae to the st. So now, we just plug back in. This is our st value. We can plug them back in and we will get the motion of the system back. So for zeta greater than 1, st comes out looking like minus zeta omega n t plus or minus square root of zeta squared minus 1 times t. And you just plug this in, and x is just e to the st. But these are just pure real values. And you'll find out that the system from initial conditions on velocity and displacement just--

[WINDS DOWN]

and dies out. Zeta equals to 1. Then, st is just minus-- you get a double root-- minus omega nt, twice. And the solution for this, I can write out the whole thing. x of t here is just some A1 plus t A2 e to the minus zeta omega n t. And again, it looks-- it's just some kind of damp, not very interesting response, no oscillations.

And then finally, zeta less than 1. And this is the only one-- this one produces oscillation. And the solution for st is plus or minus-- minus zeta omega n t, a real part, plus or minus i omega n t times the square root of 1 minus zeta squared. Now, I've turned around this zeta squared minus 1. This is now a negative number. A square root of a negative number gives me i. And now, I turn this around, so this is just a real positive number. So when you get i into this answer, what does it tell you that the solution looks like?

AUDIENCE: Sines and cosines.

J. KIM VANDIVER: Sines and cosines, right? So now, this gives you sines and cosines with a decay. This is an exponential to e to the minus zeta omega n t multiplied by a sine and a cosine. And so this is the interesting part. So most of the work of the rest of this term, we're only interested in this final solution. And what it looks like for this one-- so for zeta less than 1, x of t is some Ae to the minus zeta omega n t times a cosine omega d t-- make it d times t minus a phase angle-- come out looking like that.

And if you draw it, depends on initial conditions, so again, a positive velocity and a positive displacement. It does this, but then it dies out. It's very similar to the undamped case, except that it has this damping that causes it to die out with time. But this right here, this is still the initial slope is v0 and the initial displacement here is x0. And I'm now going to give you the exact expressions for this and we'll talk about it.

Another way of writing this then in terms of the initial conditions is this looks like x0 cosine omega d t plus v0 over omega d. So expanding this out, this result clearly has to depend on the initial displacement and on the initial velocity. Now, what's this? I keep writing this omega d. So notice in here in the solution, it's omega n times the square root of 1 minus zeta squared. So the frequency that's in here isn't exactly omega n. It's omega n altered by a bit. Omega sub d is called the damped natural frequency. And it's equal to omega n times the square root of 1 minus theta squared.

The system actually oscillates at a slightly different frequency. And for most systems that vibrate at all, this damping term is quite small. And when you square it, it gets even smaller. So this is usually a number that's 0.99, oftentimes, or even bigger than that. This is very close to 1 for all small amounts of damping. But being really careful about this in including it everywhere, that's what this result looks like. And this little thing, psi, this little phase angle here, is tangent inverse of theta over the square root of 1 minus theta squared.

And this number-- when damping is small, this is a very small number. And most of the time of problems that we deal with, the damping will be small. So let's say, for small damping-- and by that, I mean zeta, say, less than 10%, what we call 10%, 0.1. And if you have a little more-- you don't care too much about the precision, it might even be 20%. Actually, if it were 0.2, squared is 0.04, right? 1 minus 0.04-- 0.96 square root, 0.98. So even with 20% damping, the difference between the undamped natural frequency and the damped natural frequency's 2%.

So for most cases with any kind of small damping at all, we can write an approximation which is easier to remember. And it's all I carry around in my head. I can't remember this, quite frankly. Don't try to and I would instead express the answer to this as just x0 cosine omega d t plus v0 over omega d sine omega damped times time times e to the minus zeta omega n t. So why do I bother to carry the omega d's along if I just said that they're almost exactly the same. For light damping, then omega n's approximately omega d.

Well, you need to keep this one in here because even though it's only 2% difference at 20% damping, if you say the solution is omega n when it's really omega d, this thing will accumulate a phase error over time. So it's gets bigger and bigger, this error here, because you haven't taken care of that little 2%. That 2% can bite you after you go through enough cycles. So I keep omega d in the expression here.

But other than that, it's almost exactly the same expression that we just came up to for the simple response of an undamped system to initial conditions, x0 cosine plus v0 over omega n sine. And now, all we've added to it is put the transient decay and the fact that it decays into the expression and changed the frequency it oscillates at to omega d instead of omega n.

So I'm going to try to impress something on you. If I took this pendulum and my stopwatch, measured the natural frequency of this thing, I could get a very accurate value if I do it carefully. Then, I take the same object and I dunk it in water and it goes back and forth. And it conspicuously goes back and forth but dies down now after a while, because it's got that water damping it. But I measure that frequency and it's 10% different, 20% different.

And I have seen people make this mistake dozens of times. You say, that's the experiment. Explain why. What's the reason that that measured frequency has changed? Got any ocean engineers in the audience? All right. So why does-- if you put the pendulum in water-- and it's still oscillating now. So it isn't so damp that it's--

[BLOWS]

So it's got some damping. It's dying out and the natural frequency's changed by 15% or 20%. What's the explanation? And the answer you always get from people is, damping. Why? Because everybody's been taught this thing, right? And they all then assume that the change in the frequency is caused by damping. But damping couldn't possibly be the reason, because with 20% damping, this thing'll die out in about two swings and it's done. That's a lot of damping, actually, but it only accounts for 2% change in natural frequency, not 15%. Hmmm. So what causes the change in the frequency?

AUDIENCE: Buoyancy of the pendulum?

J. KIM VANDIVER: No, not buoyancy. That could actually have an effect. That's actually-- I should say, yes, you're partly right. There's another reason. When the thing is swinging back and forth there in the water, it actually carries some water with it. Effectively, the kinetic energy-- you now know how to do vibration problems. Find the equations of motion accounting for the potential energy and the kinetic energy. The kinetic energy changes, because some water moves with the object and it's called added mass.

It literally-- there is water moving with the object that has kinetic energy associated with the motion and it acts like it's more massive. It is dynamically more massive. There's water moving with it. So trying to impress on you that damping doesn't cause much of a change in systems that actually vibrate. Really observe the vibration. If you can observe the vibration, damping cannot possibly account for a very large shift in frequency.

What's the motion look like? Let's move on a little bit here. So that's what this solution looks like. We know it depends on initial conditions. The distance from here to here will make this a time axis. This is one period. So this is tau d. That's the damped period of vibration. And we know that x of t is some Ae to the minus theta omega n t cosine omega d t minus a phase angle. We can write that expression like this. And this term, this is just a cosine. This term repeats every period, right? If it's at maximum value here, exactly one period later, it's again at its maximum. So the cosine term goes to 1 every 2 pi or every period of motion, right?

So I want to take-- I'm going to define this as the value at x at some time t. I'll call it t0. And out here is x at t0 plus n tau d, n periods later. So this is the period, defined as period. Remember, omega d is the same thing as 2 pi times the frequency in hertz. And frequency is 1 over period, 2 pi over the period.

So remember, there's a relationship that you need to remember now that relates radian frequency to frequency in cycles per second in hertz to frequency expressed in period. All right? This would be tau d here and this would be an f d. For any frequency, you can say that. At omega is 2 pi f is 2 pi over tau. So you've got to be good with that.

But now, so here we are, two peaks separated by n periods. And I want to take the ratio of x of t to x of t plus n tau d here. And that's just going to be then my-- when I take that ratio, x of t has cosine omega d t minus phi in it. And n periods later, exactly the same thing appears, right? So the cosine term just cancels out. This just is e-- and the A's cancel out.

That's the initial conditions. It's e to the minus zeta omega n t-- and I guess I called it t0-- over e to the minus zeta omega n t0 plus n damped periods. And if I bring this into the numerator, the exponent becomes positive. The t0 terms, minus zeta omega and t0 plus, those cancel. And this expression is just e to the plus zeta omega n times n td.

And the last step that I want to do to this, what I'm coming up with is a way of estimating-- purposely doing this-- is this transient curve we know is controlled by a damping, by zeta. I want to have an experimental way to determine what is zeta. And I do it by computing something called the logarithmic decrement. So if I take the natural log of x of t over x of t plus n periods, it's the natural log of this expression. So I just get the exponent back. This then is n zeta omega-- I guess I better to do it carefully-- omega n n tau d. The tau d is 2 pi over omega and I get some nice things to cancel out here.

So this natural log over the ratio-- this is n zeta omega n and this is 2 pi over omega d, which is omega n times the square root of 1 minus zeta squared. Omega n's go away. And for zeta small, this term's approximately 1, in which case this then becomes n 2 pi zeta. And zeta equals 1 over 2 pi n natural log of this ratio of x of t over x of t plus nt.

So experimentally, if you just go in and measure your-- if you plot out the response, you measure a peak value, you measure the peak value n periods later, compute the log of that ratio, divide by 1 over 2 pi n, the number of periods, you have an estimate of the natural frequency-- estimate of the damping ratio, excuse me. And to give you one quick little rule of thumb here, so this is an experimental way that very quickly, you can estimate the damping of a pendulum or whatever by just doing a quick measurement.

So if it happens that after n periods, this value is half of the initial value, then this ratio is 2, right? So x of t-- some n periods later, this is only half as big. This value's 2. The natural log of 2 is some number you can calculate. So there's a little rule. If you just work that out, you find that zeta equals 1 over 2 pi n 50% times the natural log of 2. And you end up here was 0-- let me do this carefully-- 1 over 2 pi, n 50%, natural log of 2. And that is 0.11 over n 50%.

That's a really handy little engineer tool to carry around in your head. So if I have an oscillator, this little end here, I can do an experiment. Give it initial deflection and it starts off at six inches or three inches amplitude. And you let it oscillate until you see it die down to half of that value. So let's say, one, two, about four cycles this thing decays by about 50%. Four cycles-- plug 4 into that formula. You get about 0.025. Agree? 2 and a 1/2% damping. Really very convenient little thing to carry around with you-- measure pendulum, how much damping does it have?

And now, this is what I'm saying. Most things that have any substantial amount of vibration, the damping is going to be way less than 10%. If it dies, if it takes one cycle for the amplitude to decrease, one cycle for the amplitude to decrease by 50%, how much damping does it have?

AUDIENCE: 11%.

J. KIM VANDIVER: 11%. So 11% damping is a lot of damping. The thing starts out here and the next cycle, it's half gone, and the next cycle after that, it's half of that. And so in about three cycles, it's gone. So if you see anything that's vibrating any length of time at all, its damping is way less than 10% and this notion of small damping is a perfectly good one.

And I'll close by just saying one other thing. If something vibrates a lot, the damping's small. You need small damping for things to actually vibrate very much. This thing, this is vibrating--

[HIGH TONE]

that high-pitched one, that's about a kilohertz. How many cycles do you think it's gone through to get down to 50% of that initial amplitude that you could hear? A few thousand? How much damping do you think this rod has? Really tiny, really tiny.

All right. So even though all we talked about today was single degree of freedom oscillators, I hope you learned a few things that we'll carry now through the rest of the term. We'll use all these concepts that we did today to talk about more complicated vibration. Good luck on your 2.001 quiz. See you on Tuesday.