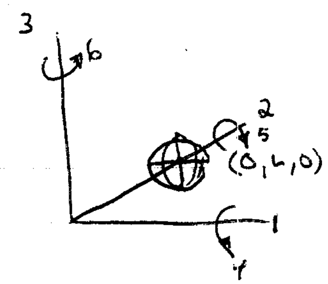
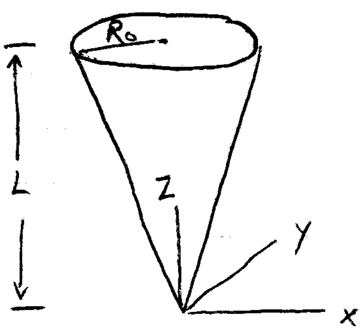
2.016 HW #4

1. sphere, volume $\forall = \rho \frac{4}{3}\pi R^3$, at $(0, L, 0)$

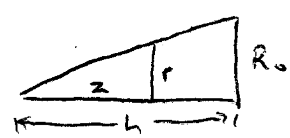


	1	2	3	4	5	6
1	X	O	O	O	O	X
2		X	O	O	O	O
3			X	X	O	O
4	Symmetric			X	O	O
5					O	O
6						X

2.



$$\forall_{cone} = \frac{1}{3}\pi R_o^2 L$$

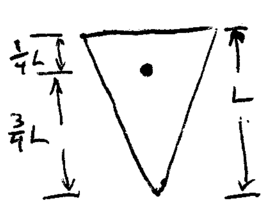


$$r = R_o \frac{z}{L}$$

a) Assume the entire body is submerged. center of buoyancy is at the center of mass:

$$\bar{z} = \frac{\int z\, dm}{\int dm} = \frac{\int_0^L z\, \rho \pi r^2\, dz}{\rho \forall_{cone}} = \frac{\int_0^L z \rho \pi \frac{R_o^2}{L^2} z^2\, dz}{\rho \frac{1}{3}\pi R_o^2 L}$$

$$= \frac{\cancel{\rho \pi \frac{R_o^2}{L^2}} \cdot \left(\frac{1}{4} L^4\right)}{\cancel{\rho \frac{1}{3}\pi R_o^2} L}$$



$$\boxed{\bar{z} = \frac{3}{4} L}$$

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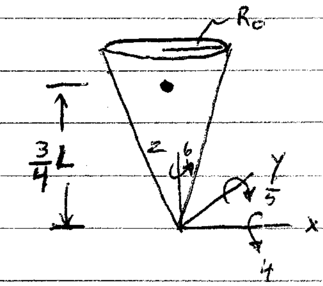
2b.



	1	2	3	4	5	6
1	X	0	0	0	X	0
2	symmetric	X	0	X	0	0
3			X	0	0	0
4				X	0	0
5					X	0
6						0

c.
$$M_{11} = M_{22} = \int_0^L \rho\pi r^2\, dz = \int_0^L \rho\pi\left(R_o \frac{z}{L}\right)^2 dz = \underline{\rho\tfrac{1}{3}\pi R_o^2 L}$$

$$M_{55} = M_{44} = \int_0^L \rho\pi r^2 \cdot z^2 \cdot dz = \int_0^L \rho\pi\left(R_o \frac{z}{L}\right)^2 z^2\, dz = \rho\pi \frac{R_o^2}{L^2}\cdot\frac{1}{5}L^5 = \underline{\rho\tfrac{1}{5}\pi R_o^2 L^3}$$

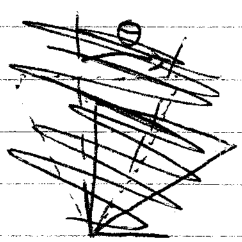
$$M_{66} = 0$$

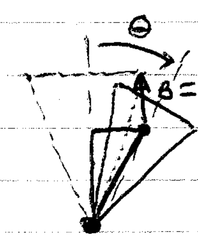
$$M_{51} = \int_0^L \rho\pi r^2 \cdot z \cdot dz = \int_0^L \rho\pi\left(R_o \frac{z}{L}\right)^2 z\, dz = \rho\pi \frac{R_o^2}{L^2}\cdot\frac{1}{4}L^4 = \underline{\rho\tfrac{1}{4}\pi R_o^2 L^2}$$

Note: All three "added mass" terms have <u>different units</u>!

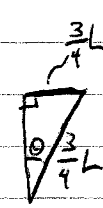
Take a look at the force equation and see if this makes sense.

d)





$B = \rho\forall g = \rho\frac{1}{3}\pi R_o^2 L g$



The cone is fixed on a hinge at the bottom.

As it tilts, the Buoyancy force creates a restoring moment.

$$M = B\cdot\tfrac{3}{4}L\theta$$

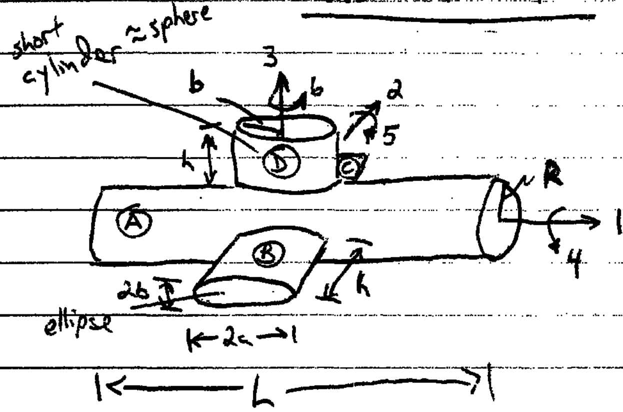
$$\boxed{(\cancel{M}+M_a)\ddot{\theta} + \left(\rho\tfrac{1}{3}\pi R_o^2 L g\right)\cdot\tfrac{3}{4}L\,\theta = 0}$$

$M \to 0$, $M_a = M_{55} = \rho\frac{1}{5}\pi R_o^2 L^3$

$$\boxed{\omega} = \sqrt{\frac{\rho\frac{1}{3}\pi R^2 L\cdot\frac{3}{4}Lg}{\rho\frac{1}{5}\pi R^2 L^3}} = \boxed{\sqrt{\frac{5}{4}\frac{g}{L}}}$$

2.016 HW #4

3.



- ignore longitudinal added mass
- ignore interactions between members
- sail is a short cylinder, so treat it as a sphere of radius b

a) $M_{33} = ?$

A cylinder $M_{33} = \rho \pi R^2 L$

B & C wings $M_{33} = 2(\rho \pi a^2 h)$

D sphere $M_{33} = \rho \frac{2}{3} \pi b^3$

$$\boxed{M_{33} = \rho \pi R^2 L + 2 \rho \pi a^2 h + \rho \frac{2}{3} \pi b^3}$$

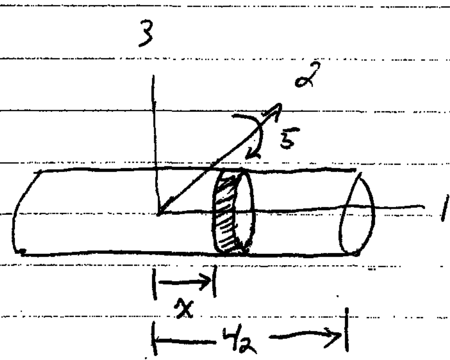
b) $M_{35} = ?$

A $M_{35} = 0$

B & C $M_{35} = 0$

D $M_{35} = 0$

$$\boxed{M_{35} = 0}$$



c) $M_{55} = ?$

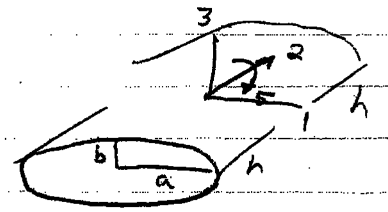
A $M_{55} = 2 \cdot \int_0^{L/2} \rho \pi R^2 \cdot x^2 \, dx$

$= 2 \rho \pi R^2 \frac{1}{3} \frac{L^3}{8}$

$= \rho \pi R^2 \frac{1}{12} L^3$

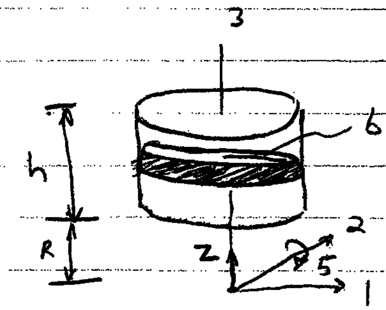
2.016 Hydrodynamics HW #4

3c. $M_{55} = ?$



(B&C) $$M_{55} = \frac{1}{8}\pi\rho\left(a^2-b^2\right)^2 2h$$

$$= \frac{1}{4}\pi\rho\left(a^2-b^2\right)^2 \cdot h$$

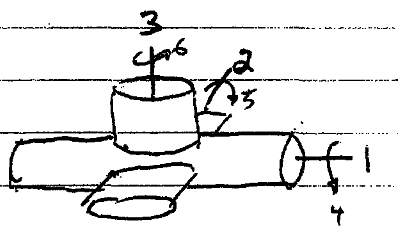


(D) $$M_{55} = \int_R^{R+h} \rho\pi b^2 \cdot z^2 \, dz$$

$$= \rho\pi b^2 \frac{1}{3}\left((R+h)^3 - R^3\right)$$

$$\boxed{M_{55} = \rho\pi R^2 \frac{1}{12} L^3 + \frac{1}{4}\pi\rho\left(a^2-b^2\right)^2 h + \rho\pi b^2 \cdot \frac{1}{3}\left((R+h)^3 - R^3\right)}$$

d)



$\vec{U} = [1,2,3,1,2,3]$

$\vec{a} = [3,2,1,3,2,1]$

	1	2	3	4	5	6
1	M_{11}	0	0	0	M_{15}	0
2		M_{22}	0	M_{24}	0	0
3			M_{33}	0	0	0
4	symmetric			M_{44}	0	0
5					M_{55}	0
6						M_{66}

$$F_1 = -\dot{U}_1 M_{11} - \dot{U}_5 M_{51} - \varepsilon_{123} U_3 U_5 M_{33} - \varepsilon_{132} U_2 U_6 M_{22} - \varepsilon_{132} U_4 U_6 M_{24}$$

$$F_1 = -3M_{11} - 2M_{51} - 1\cdot 3\cdot 2\, M_{33} - (-1)\cdot 2\cdot 3\, M_{22} - (-1)(1)(3)\, M_{24}$$

$$F_2 = -\dot{U}_i M_{2i} - U_i U_6 M_{1i} + U_i U_4 M_{3i}$$

$$= -\dot{U}_2 M_{22} - \dot{U}_4 M_{24} - U_1 U_6 M_{11} - U_5 U_6 M_{15} + U_3 U_4 M_{33}$$

$$F_2 = -2M_{22} - 3M_{24} - 3M_{11} - 6M_{15} + 3M_{33}$$

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$$F_3 = -\dot{U}_i M_{3i} - U_i U_4 M_{2i} + U_i U_5 M_{1i}$$

$$= -\dot{U}_3 M_{33} - U_2 U_4 M_{22} - U_4 U_4 M_{24} + U_1 U_5 M_{11} + U_5 U_5 M_{15}$$

$$F_4 = M_1 = -\dot{U}_i M_{4i} - U_i U_5 M_{6i} - U_i U_2 M_{3i} + U_i U_6 M_{5i} + U_i U_3 M_{2i}$$

$$= -\dot{U}_2 M_{42} - \dot{U}_4 M_{44} - U_6 U_5 M_{66} - U_3 U_2 M_{33} + U_1 U_6 M_{51} + U_5 U_6 M_{55} + U_2 U_3 M_{22} + U_4 U_3 M_{24}$$

$$F_5 = M_2 = -\dot{U}_i M_{5i} - U_i U_6 M_{4i} - U_i U_3 M_{1i} + U_i U_4 M_{6i} + U_i U_1 M_{3i}$$

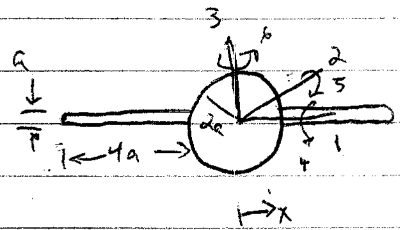
$$= -\dot{U}_1 M_{51} - \dot{U}_5 M_{55} - U_2 U_6 M_{42} - U_4 U_6 M_{44} - U_1 U_3 M_{11} - U_5 U_3 M_{15} + U_6 U_4 M_{66} + U_3 U_1 M_{33}$$

$$F_6 = M_3 = -\dot{U}_i M_{6i} - U_i U_4 M_{5i} - U_i U_1 M_{2i} + U_i U_5 M_{4i} + U_i U_2 M_{1i}$$

$$= -\dot{U}_6 M_{66} - U_1 U_4 M_{51} - U_5 U_4 M_{55} - U_2 U_1 M_{22} - U_4 U_1 M_{24} + U_2 U_5 M_{42} + U_4 U_5 M_{44} + U_1 U_2 M_{11} + U_5 U_2 M_{15}$$

2.016 HW#4

4.



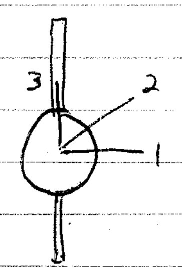
	1	2	3	4	5	6
1	$\rho\frac{2}{3}\pi a^3$	0	0	0	0	0
2		X	0	0	0	0
3			X	0	0	0
4				0	0	0
5					X	0
6						X

$$M_{11} = \rho\frac{2}{3}\pi(2a)^3$$

$$M_{22} = M_{33} = \rho\frac{2}{3}\pi(2a)^3 + \rho\pi\left(\frac{a}{2}\right)^2\cdot(8a)$$

$$M_{55} = M_{66} = \int_a^{5a}\rho\pi\left(\frac{a}{2}\right)^2 x^2\,dx = \rho\pi\frac{a^2}{4}\cdot\frac{1}{3}\cdot\left(125a^3 - a^3\right) = \rho\pi\frac{124}{12}a^5$$

For the vertical orientation, just rotate the coordinate system



$$M_{11} = M_{22} = \rho\frac{2}{3}\pi(2a)^3 + \rho\pi\,2a^3$$

$$M_{33} = \rho\frac{2}{3}\pi(2a)^3$$

$$M_{55} = M_{11} = \rho\pi\frac{124}{12}a^5$$

a) $B = \rho\forall g$

$$F_3 = -\dot{U}_3 M_{33} = \begin{cases} \rho\frac{2}{3}\pi(2a)^3 + \rho\pi\,2a^3 = \frac{22}{3}\rho\pi a^3 & \text{(horizontal)} \\ \rho\frac{2}{3}\pi(2a)^3 = \frac{16}{3}\rho\pi a^3 & \text{(vertical)} \end{cases}$$

$\dot{U}_3$

$$mg = \cancel{\rho_s}^{\,2\rho}\forall g = 2\rho\left(\frac{4}{3}\pi(2a)^3 + \pi\left(\frac{a}{2}\right)^2\cdot 8a\right)g = \frac{76}{3}\rho\pi a^3 g$$

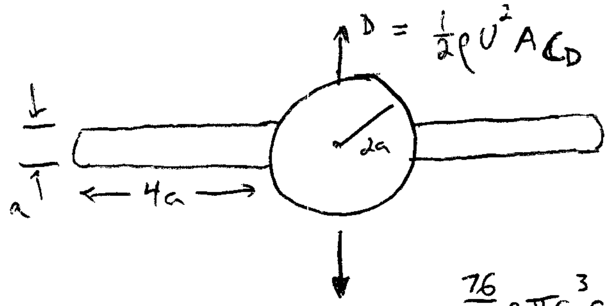
$$\Sigma\vec{F} = m\vec{a} = -m\dot{U}_3\hat{k} = -mg\hat{k} + \dot{U}_3 M_{33}\hat{k} + \rho\forall g\hat{k}$$

$$\boxed{\dot{U}_3 = \frac{m-\rho\forall}{m+M_3}g} = \begin{cases} \dfrac{\frac{76}{3}\rho\pi a^3 - \frac{38}{3}\rho\pi a^3}{\frac{76+22}{3}\rho\pi a^3}g = \boxed{\frac{38}{98}g} & \text{(horizontal)} \\ \dfrac{\frac{76}{3}\rho\pi a^3 - \frac{38}{3}\rho\pi a^3}{\frac{76+16}{3}\rho\pi a^3}g = \boxed{\frac{38}{92}g} & \text{(vertical)} \end{cases}$$

2.016 HW#4

4b.

$D = \frac{1}{2}\rho U^2 A C_D$



$mg = \frac{76}{3}\rho \pi a^3 g$

$D = mg$ at terminal velocity

$$U_{terminal} = \left(\frac{\frac{76}{3}\rho\pi a^3 g}{\Sigma \frac{1}{2}\rho A C_D}\right)^{1/2}$$

$\frac{76}{3} \approx 25$

Drag for cylinders: $A = 2(4a \cdot a) = 8a^2$

$C_D = 1.2$

$\frac{1}{2}\rho A C_D = \frac{1}{2}\rho\, 8a^2 \cdot 1.2 \approx 4\rho a^2$

for sphere $A = \pi(2a)^2 = 4\pi a^2 \approx 12a^2$

$C_D = 0.5$

$\frac{1}{2}\rho A C_D = \frac{1}{2}\rho \cdot 12a^2 \cdot 0.5 = 3\rho a^2$

$$U_{terminal} \approx \left(\frac{25\cancel{\rho}\pi a^{\cancel{3}} g}{7\cancel{\rho}\cancel{a^2}}\right)^{1/2} \approx (10ag)^{1/2} \quad \text{for large } a$$

for small a, ignore the drag from the cylinders

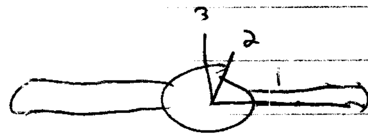
$$U_{terminal} \approx \left(\frac{25\cancel{\rho}\cancel{\pi}a^{\cancel{3}} g}{\cancel{3}\cancel{\rho}\cancel{a^2}}\right)^{1/2} \approx (25ag)^{1/2} \quad \text{for small } a.$$

(In reality C_D depends on U, a, and the viscosity of water, but we assume these numbers here in order to solve this problem...)

2.016 HW #4

$$M_j = -\dot{U}_i M_{j+3,i} - \varepsilon_{jkl} U_i \Omega_k M_{l+3,i} - \varepsilon_{jkl} U_k U_i M_{li}$$

4c.



$$\vec{U} = (0, -U_2, U_3, 0, 0, 0)$$

	1	2	3	4	5	6
1	x	0	0	0	0	0
2	0	x	0	0	0	0
3	0	0	x	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	x	0
6	0	0	0	0	0	x

$$M_1 = -\dot{U}_i M_{4i} - U_2 U_i M_{3i} + U_3 U_i M_{2i}$$

$$= -U_2 U_3 M_{33} + U_3 U_2 M_{23}$$

$$\boxed{M_1 = U_2 U_3 M_{33} - U_2 U_3 M_{23}}$$

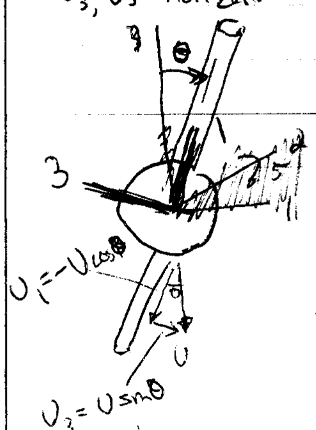
$$M_2 = -\dot{U}_i M_{5i} - \overset{1}{\varepsilon_{231}} U_3 U_i M_{1i} - \overset{(-1)}{\varepsilon_{213}} \overset{0}{\phi_1} U_i M_{3i}$$

$$\boxed{M_2 = 0}$$

$$M_3 = -\overset{0}{\dot{U}_i} M_{6i} - \varepsilon_{312} \overset{0}{\phi_1} U_i M_{2i} - \varepsilon_{321} U_2 U_i M_{1i}$$

$$\boxed{M_3 = 0}$$

d. U_3, U_5 nonzero



$$M_2 = -\varepsilon_{jkl} U_k U_i M_{li} = -\varepsilon_{213} U_1 U_i M_{3i} - \varepsilon_{231} U_3 U_i M_{1i}$$

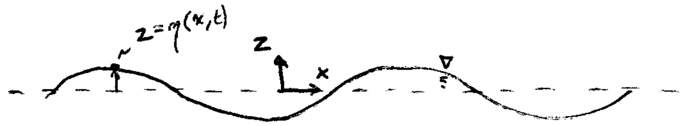
$$= U_1 U_3 M_{33} - U_3 U_1 M_{11}$$

$$\boxed{M_2 = U_1 U_3 (M_{33} - M_{11})}$$

$$\boxed{M_2 = -U^2 \sin\theta \cos\theta (M_{33} - M_{11})}$$

2.016 HW#4

5. 2D Deep water wave



Free surface at $z = \boxed{\eta(x,t) = a\cos(kx-\omega t)}$

velocity potential $\boxed{\phi(x,z,t) = + \frac{a\omega}{k} e^{kz}\sin(kx-\omega t)}$

$a = 2.5\text{ ft} = 0.762\text{ m}$

$\lambda = 120\text{ ft} = 36.6\text{ m} \longrightarrow k = \frac{2\pi}{\lambda} = 0.172\ \text{1/m}$

$\omega^2 = gk \longrightarrow \omega = 1.30\ \text{1/s}$

$z_n = \frac{\lambda}{8} = 4.58\text{ m}$

a) $P_{dyn} = -\rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right] = -\rho\left[\underbrace{\frac{-a\omega^2}{k}}_{=ag} e^{kz}\cos(kx-\omega t) + \frac{1}{2}a^2\omega^2 e^{2kz}\underbrace{(\cos^2(kx-\omega t)+\sin^2(kx-\omega t))}_{=1}\right]$

$$\boxed{P_{dyn} = -\rho a g e^{kz}\cos(kx-\omega t) + \cancel{\frac{1}{2}\rho a^2\omega^2 e^{2kz}}}$$

If you look at the size of this compared to the other term, you see it is small and can be ignored! (the \$1000 vs \$1 argument)

$$\boxed{\begin{aligned} u &= \frac{\partial\phi}{\partial x} = +a\omega e^{kz}\cos(kx-\omega t) \\ a_x &= \frac{\partial u}{\partial t} = +a\omega^2 e^{kz}\sin(kx-\omega t) \\ w &= \frac{\partial\phi}{\partial z} = +a\omega e^{kz}\sin(kx-\omega t) \\ a_z &= \frac{\partial w}{\partial t} = -a\omega^2 e^{kz}\cos(kx-\omega t)\end{aligned}}$$

see plots in Excel for $0 \le x \le 2\lambda$, $t=0$

b) see plots for $x=0$, $0 \le t \le 2\cdot\left(\frac{2\pi}{\omega}\right)$

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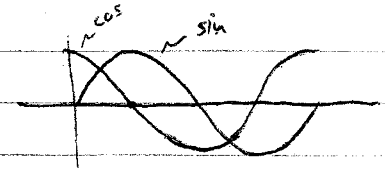
5c. $\eta = a\cos(kx-\omega t)$



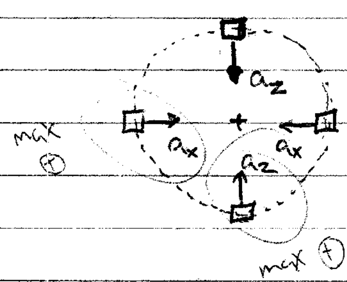
$a_x = -a\omega^2 e^{kz}\sin(kx-\omega t)$

sine is maximum when cosine is zero, so a_x is maximum at a nodal point.

$a_z = a\omega^2 e^{kz}\cos(kx-\omega t)$

cosine is maximum when cosine is one, so a_z is maximum positive (up) at a wave trough.

Remember, Bob travels in a circular orbit (pathline)



$\eta \propto \cos(kx-\omega t)$

d. $u \propto -\cos(kx-\omega t)$

$\hookrightarrow u^2$ max at wave crest or trough

$w \propto -\sin(kx-\omega t)$

$\hookrightarrow w^2$ max at nodal point.

e. $P_{Total} = P_{dynamic} + P_{Hydrostatic} = \underbrace{\rho a g e^{kz}\cos(kx-\omega t)}_{\text{from part (a)}} + P_{atm} - \rho g(z-\eta)$

at crest $\eta = a$, nodal point $\eta = 0$, trough $\eta = -a$

z is measured from the ~~mean~~ average height