28 Floating Structure in Waves

We consider the pitch and heave dynamics of a large floating structure in a random sea. You can consider this a two-dimensional problem.

The structure has two main, identical struts that pierce the water: each has area A_w of two hundred square meters, and their centers are separated by a distance L of fifty meters. The mass center of the structure is at the mid-point. The mass m is 1000 tons, and the mass moment of inertia about the centroid is $J = 4.0 \times 10^5 ton \cdot m^2$. Each hull has an apparent linear damping in the vertical direction of $b = 60kN \cdot s/m$.

The horizontal motion of the structure is nearly zero. The vertical excitation force exerted at each of the struts may be approximated as the stiffness (provided by the strut's water-plane area) times $\eta - \zeta$, where η is the wave elevation at the location of the strut's center, and ζ is the vertical displacement of the strut. Make linearizations where needed. Note we do not take into account any added mass forces in this problem. Also, we assume that the mass center is low on the water, so that the pitching moment is generated exactly by the net loss of flotation on one side and/or the increase of flotation on the other.

For the wave description, we use the Bretschneider spectrum; it is given by

$$\begin{aligned} S(\omega) &= \frac{A}{\omega^5} e^{-B/\omega^4}, \text{ where} \\ \omega_m &= \text{modal (or peak) frequency, rad/s} \\ B &= 1.25\omega_m^4; \qquad A = 4BE_S; \qquad E_S = H_{1/3}^2/16. \end{aligned}$$

In SeaState 5, we take the modal period as 9.7 seconds, and the significant wave height $H_{1/3}$ as 3.3m. We assume that the waves are all traveling in the same direction, from negative x toward positive x.

1. Write a pair of differential equations, that express the heave motion of the center of mass (say $z(t)$), and the pitch motion (say $\phi(t)$), in terms of the wave elevations at the two struts. Hint: use the fact that

$$\zeta(t, -L/2) = z(t) - \phi(t)L/2, \quad \text{and so on.}$$

Solution: We have, using the hint,

$$\begin{aligned} m\ddot{z} &= \rho g A_w[\eta(-L/2) - (z - L\phi/2) + \eta(L/2) - (z + L\phi/2)] - \\ &\quad b[(\dot{z} - L\dot{\phi}/2) + (\dot{z} + L\dot{\phi}/2)] \\ &= \rho g A_w[\eta(-L/2) + \eta(L/2) - 2z] - 2b\dot{z} \\ J\ddot{\phi} &= \rho g A_w \frac{L}{2}[-\eta(-L/2) + (z - L\phi/2) + \eta(L/2) - (z + L\phi/2)] + \\ &\quad \frac{L}{2} b[(\dot{z} - L\dot{\phi}/2) - (\dot{z} + L\dot{\phi}/2)] \\ &= \rho g A_w \frac{L}{2}[\eta(L/2) - \eta(-L/2) - L\phi] - \frac{L^2}{2} b\dot{\phi}. \end{aligned}$$

Because of cancelations in this symmetric situation, we end up with decoupled equations - i.e., the pitch motion does not affect the heave motion, and vice versa.

2. What are the structure's natural frequencies in heave and in pitch? Do these seem like a good design?

   Solution: The natural frequency in heave, obtained from the above equation with no wave excitation ($\eta = 0$) is $\sqrt{2\rho g A_w/m} = 1.98 rad/s$, or a period of $3.2 sec$. The natural frequency in pitch is $\sqrt{\rho g A_w L^2/2J} = 2.48 rad/s$, or a period of $2.5 sec$. These are quite fast compared to the frequencies that we expect in big seas - a three-second wave in the open ocean is generally quite small in amplitude, less than one meter. So it seems like a good design from the resonance point of view.

3. Give a general expression for the wave elevation at one strut, as a function of the wave elevation at the other strut. To do this, write down the elevations for only one wave, at frequency ω. You will use the dispersion relation to work out the wavelength λ, and hence derive a frequency-dependent phase angle.

   Solution: We have

$$\begin{aligned}\eta(L/2) &= \eta(-L/2)\cos\left(\frac{2\pi L}{\lambda}\right)\\ &= \eta(-L/2)\cos\left(\frac{L\omega^2}{g}\right),\end{aligned}$$

   where the second expression is found by substituting the dispersion relation.

4. Insert this result into your set of differential equations, and come up with an ODE for the heave being driven by waves (referenced to $x = -L/2$), and another for the pitch being driven by waves. Note your answers will have a "weird" term similar to $\cos(\omega^2)$, which you can carry directly into the frequency domain (because the Fourier transform is an integration over time!).

   Solution:

$$\begin{aligned}m\ddot{z} + 2b\dot{z} + 2\rho g A_w z &= \rho g A_w(1 + \cos(L\omega^2/g))\eta(-L/2)\\ J\ddot{\phi} + \frac{L^2}{2}b\dot{\phi} + \frac{L^2}{2}\rho g A_w \phi &= -\frac{L}{2}\rho g A_w(1 - \cos(L\omega^2/g))\eta(-L/2).\end{aligned}$$

5. What are the two transfer functions

$$\frac{z(j\omega)}{\eta(j\omega, x = -L/2)} \quad \text{and} \quad \frac{\phi(j\omega)}{\eta(j\omega, x = -L/2)}\ ?$$

   Solution:

$$\begin{aligned}\frac{z(j\omega)}{\eta(j\omega, x = -L/2)} &= \frac{\rho g A_w[1 + \cos(L\omega^2/g)]}{-m\omega^2 + 2bj\omega + 2\rho g A_w}\\ \frac{\phi(j\omega)}{\eta(j\omega, x = -L/2)} &= -\frac{\frac{L}{2}\rho g A_w[1 - \cos(L\omega^2/g)]}{-J\omega^2 + \frac{L^2}{2}bj\omega + \frac{L^2}{2}\rho g A_w}.\end{aligned}$$

6. Make a labeled plot of the Bretschneider wave spectrum for these SS5 conditions.

   See the attached graph.

7. What are the significant heights (double amplitudes) of the heave and the pitch motions? Based on plots of the output spectra, about what are the dominant frequencies of these two motions?

   Solution: The significant heave height is 3.30m and the significant pitch height is 0.121rad. The output spectra plots (attached) are interesting because the dominant frequency in the heave direction is close to the resonance, around 1.9rad/s, or 3.3 seconds period. On the other hand, the pitch direction has a dominant frequency much lower, at about 0.75rad/s, or 8.4 seconds period - this is closer to the excitation frequency. You see that the effect of the different wavelengths gives rise to many peaks in the responses, and this ultimately controls the output spectra shapes.

8. What are the heave and pitch amplitudes expected to be exceeded in ten minutes? one hour? one day?

   Solution: For heave the amplitudes that will be exceeded are [2.63 3.06 3.70] meters; for pitch, we find [0.094 0.110 0.134] radians. The formulas are (for the z part):

$$\begin{aligned}
M_{iz} &= \int_0^\infty \omega^i S_z(\omega)d\omega \\
\bar{T}_z &= 2\pi\sqrt{\frac{M_{0z}}{M_{2z}}} \\
\bar{f}(A_z) &= [1/600 \quad 1/3600 \quad 1/86400] \;\; Hertz \\
A_z &= \sqrt{-2M_{0z}\log\left(\bar{f}(A_z)\bar{T}_z\right)}.
\end{aligned}$$

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%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Heave and pitch response of a large floating structure

clear all;

wm = 2*pi/9.7 ; % modal frequency of waves, rad/s
Hsig = 3.3 ; % significant wave height, m
wvec = 0.001:0.001:4 ; % vector of frequencies to consider

% make up the Bretschneider spectrum
for j = 1:length(wvec),
    w = wvec(j) ;
    S(j) = 5/16 * wm^4 / w^5 * Hsig^2 * exp(-5 * wm^4 / 4 / w^4) ;
end;
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% check that we got the right formula!
disp(sprintf(...
    'Square Root of Integral of Area of S:  %g;  Hsig/4:  %g', ...
    sqrt(sum(S)*mean(diff(wvec))), 1/4*Hsig));

% plot the spectrum
figure(1);clf;hold off;
subplot(211);
plot(wvec,S,'LineWidth',2);
grid;
title('Sea Wave Spectra for Sea State 5');
xlabel('frequency \omega, rad/s');
ylabel('S(\omega)');

print -deps bigStructure.eps

% give the physical parameters of the structure
L = 50 ; % distance between flotation centers, m
m = 1e6 ; % material mass, kg
J = 4e8 ; % material rotary moment of inertia, kg-m^2
Aw = 200 ; % waterplane area per hull, m^2
rho = 1000 ; % water density, kg/m^3
g = 9.81 ; % gravity, m/s^2
b = 60000 ; % linear damping coefficient in vertical direction,
            % per hull, N/(m/s)

% compute the numerical transfer functions from wave elevation at
% x=-L/2 to height (z) and pitch angle (phi).  The units are
% meter/meter and rad/meter
for i = 1:length(wvec),
    w = wvec(i) ;
    eta2z(i) = (1 + cos(L*w*w/g))*rho*g*Aw / ...
        (-m*w^2 + sqrt(-1)*w*2*b + 2*g*Aw*rho) ;
    eta2phi(i) = -(1 - cos(L*w*w/g))*rho*g*Aw*L/2 / ...
        (-J*w^2 + sqrt(-1)*w*L^2/2*b + rho*g*Aw*L^2/2) ;
end;

% (undamped) natural frequencies
wnz = sqrt(2*g*Aw*rho/m) ;
wnphi = sqrt(rho*g*Aw*L^2/2/J) ;
disp(sprintf('Natural frequencies:  %g rad/s (heave)', wnz));
disp(sprintf('                      %g rad/s (pitch)', wnphi));

% get the spectra of the heave motion and the pitch angle.
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% (Take only the real part because there are some parasitic,
% tiny imaginary parts that are annoying.)
Sz = real(S.*eta2z.*conj(eta2z)) ;
Sphi = real(S.*eta2phi.*conj(eta2phi)) ;

% plot these spectra
figure(2);clf;hold off;
subplot(211);
plot(wvec,Sz,'LineWidth',2);
grid;
ylabel('S_z, m^2s');
subplot(212);
plot(wvec,Sphi,'LineWidth',2);
grid;
ylabel('S_{\phi}, rad^2 s');
xlabel('frequency, rad/s');
print -deps bigStructure2.eps

% compute the significant heights of the heave and the pitch motions
zsig = 4*sqrt(sum(Sz)*mean(diff(wvec))) ;
phisig = 4*sqrt(sum(Sphi)*mean(diff(wvec))) ;
disp(sprintf('Significant z height:  %g m', zsig));
disp(sprintf('Significant phi height:  %g rad',phisig));

% compute the ten-minute, one-hour, and one-day extreme amplitudes
M0z = sum(Sz)*mean(diff(wvec)); % moments of the spectra
M2z = sum(Sz.*wvec.^2)*mean(diff(wvec));
M0phi = sum(Sphi)*mean(diff(wvec));
M2phi = sum(Sphi.*wvec.^2)*mean(diff(wvec));

Tbarz = 2*pi*sqrt(M0z/M2z) ; % average period
Tbarphi = 2*pi*sqrt(M0phi/M2phi) ;

fbarA = [1 1 1]./[10*60  60*60  24*60*60] ; % event frequencies, Hz

Az = sqrt(-2*zsig^2/16*log(Tbarz*fbarA));
Aphi = sqrt(-2*phisig^2/16*log(Tbarphi*fbarA));

disp('Ten-minute, one-hour, and one-day amplitudes:');
disp(sprintf('Heave:  %g  %g  %g m', Az(1),Az(2),Az(3)));
disp(sprintf('Pitch:  %g  %g  %g rad', Aphi(1),Aphi(2),Aphi(3)));

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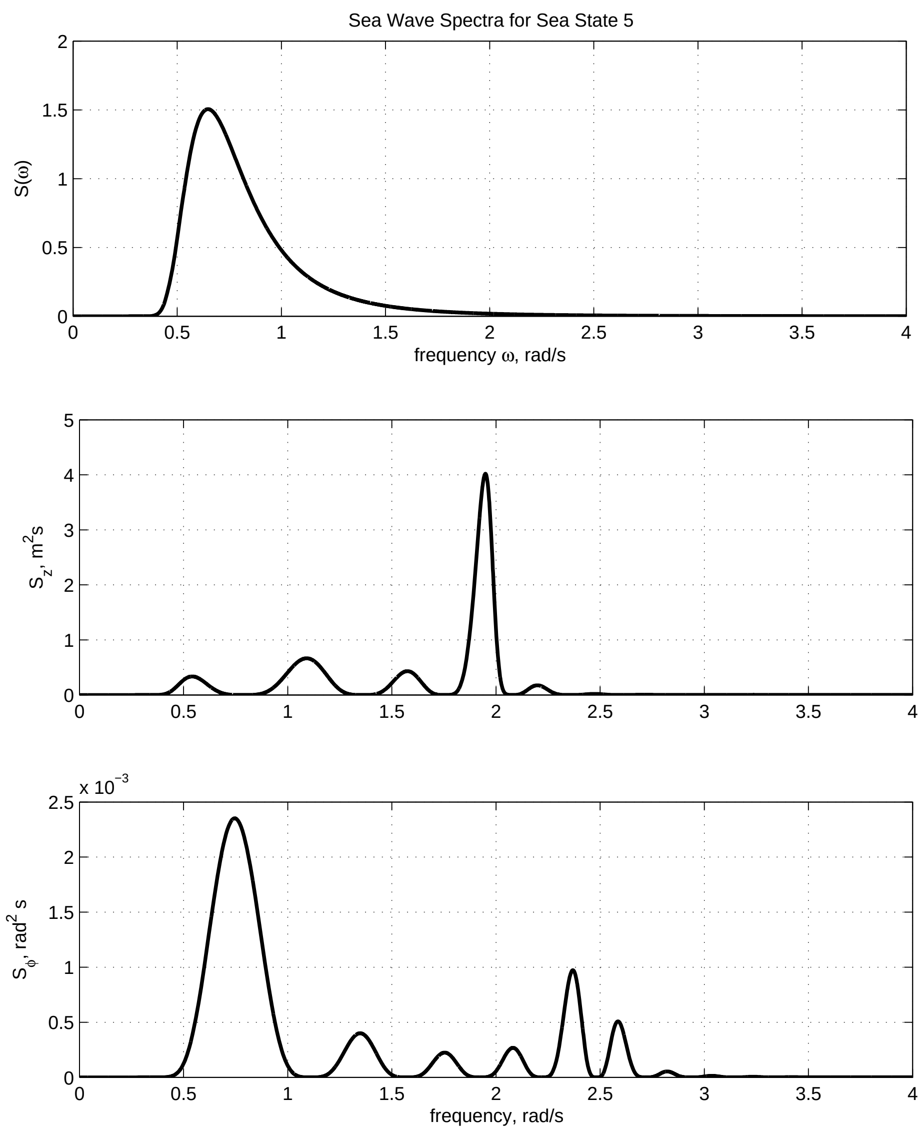
Sea Wave Spectra for Sea State 5
S(ω)
frequency ω, rad/s
S_z, m^2s
x 10^-3
S_φ, rad^2 s
frequency, rad/s

2.017J Design of Electromechanical Robotic Systems
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