1.685J/2.034J/18.377J Nonlinear Dynamics and Waves Spring 2007

Problem Set No. 1

Out: Thursday, February 15, 2007

Due: Thursday, March 1, 2007 *in class*

Problem 1

A man rows a boat across a river of width a occupying the region $0 \leq x \leq a$ in the x, y-plane, always rowing towards a fixed point on one bank, say (0,0). He rows at constant speed u relative to the water, and the river flows at constant speed v. Show that

$$\dot{x} = \frac{-ux}{(x^2 + y^2)^{1/2}}, \qquad \dot{y} = v - \frac{uy}{(x^2 + y^2)^{1/2}},$$

where (x, y) are the coordinates of the boat. Show that the phase trajectories are given by

$$y + (x^2 + y^2)^{1/2} = Cx^{1-\alpha},$$

where $\alpha = v/u$. Sketch the phase diagram for $\alpha < 1$ and interpret it. What kind of point is the origin? What happens to the boat if $\alpha > 1$?

Problem 2

Consider the system

$$\frac{dx}{dt} = y - x^3, \quad \frac{dy}{dt} = -x^3.$$

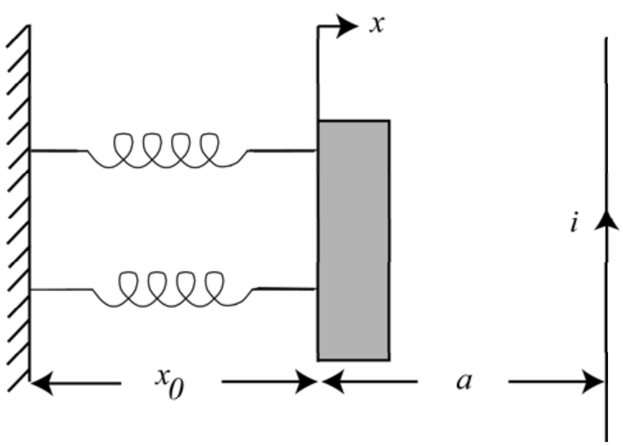
Discuss the stability of the equilibrium point at the origin (i) by linearized theory and (ii) for the full nonlinear system. (*Hint:* consider $x^4 + 2y^2$)

Problem 3

The equation of motion of a bar restrained by springs and attracted by a parallel current-carrying conductor is

$$\ddot{x} + c\left\{x - \frac{\lambda}{a - x}\right\} = 0$$

where c, a and λ are positive constants. Sketch the phase trajectories for $-x_0 < x < a$ and classify all equilibrium points for $\lambda > 0$.



Problem 4

Consider the system governed by

$$\ddot{x} + \mu \sin \dot{x} + x = 0$$

(a) Construct several trajectories and show that more than one limit cycle exists. (You may find it useful to use the computer for this purpose.)

(b) Some limit cycles are stable while others are unstable. How can one determine the stability of the various limit cycles by examining the trajectories in the phase plane?

Problem 5

Consider the modified Van der Pol oscillator

$$\ddot{x} - \varepsilon(1 - x^4)\dot{x} + x = 0.$$

(a) For almost sinusoidal vibrations, i.e., $0 < \varepsilon \ll 1$, find the value of the dimensionless amplitude x_0 to which the system ultimately will go in a steady state.

(b) With the help of the computer, examine the range of validity of he approximate solution found in (a) as ε is increased. What happens for $\varepsilon \gg 1$

Problem 6

In a simple model of national economy,

$$\dot{I} = I - \alpha C, \;\; \dot{C} = \beta(I - C - G),$$

where I is the national income, C is the rate of consumer spending, and G the rate of government expenditure; the constants α, β satisfy $1 < \alpha < \infty, \;\; 1 \leq \beta < \infty$. Show that if the rate of government expenditure G_0 is constant, there is an equilibrium state. Classify the equilibrium state and show that the economy oscillates when $\beta = 1$.

Consider the situation when the government expenditure is related to the national income by the rule $G = G_0 + kI \;\; (k > 0)$. Show that there is no equilibrium state if $k \geq (\alpha - 1)/\alpha$. How does the economy then behave?

Discuss an economy in which $G = G_0 + kI^2$, and show that there are two equilibrium states.