

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Department of Mechanical Engineering  
**2.04A Systems and Controls**  
Spring 2013

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Practice Problems #2

**Please do not turn in – practice only**

Posted: Thursday, March 14, '13

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**1. Review questions** The purpose of the following questions is to remind you of some fundamental topics and results that we derived in this class. Answer as accurately and succinctly as possible.

**1.a)** What is the relationship between the torque constant  $K_m$  and back-emf constant  $K_v$  of a DC motor? What is the physical justification for this relationship?

Answer:  $K_m = K_v$ . This is because energy must be conserved when it is converted from the electrical to the mechanical domain by the DC Motor. It also follows from the derivation of the electromagnetic derivation of torque and back-emf in the motor coils.

**1.b)** In what sense are a spring and a capacitor analogous as elements in electro-mechanical systems?

Answer: Both spring and capacitor are energy storage elements (the spring stores elastic energy, the capacitor stores charge, *i.e.* electrostatic energy.)

**1.c)** Which parameter specifies whether a 2nd-order system is overdamped/critically damped/underdamped/undamped? How are the step responses of these systems different?

Answer: The damping ratio  $\zeta$  specifies the 2nd-order system behavior. If  $\zeta > 1$ , the system is overdamped and there is no overshoot. If  $\zeta = 1$ , then system is critically damped. If  $\zeta < 1$ , then the system is underdamped and the step response is oscillation with exponential decay envelope. If  $\zeta = 0$ , then the system is undamped. The step response is pure oscillation without any decay.

**1.d)** If a DC motor is driven to spin faster, why does it get hotter?

Answer: Higher torque will require higher current, and therefore more of the input energy will be converted into heat (inevitably, the motor wires have a finite resistance.)

- 1.e) In the Eddy braking experiments we carried out in the laboratory, we used a flywheel made out of copper. If the flywheel were made out of steel, would the braking time constant increase or decrease?

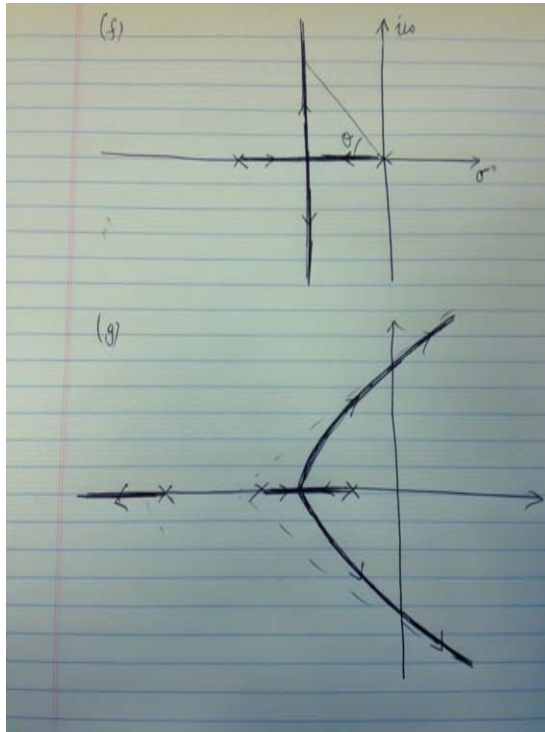
Answer: Copper has much higher conductivity than steel (copper is  $\sim 5 \times 10^7 \Omega^{-1}/\text{m}$  whereas steel is  $\sim 5 \times 10^6 \Omega^{-1}/\text{m}$  or less for various types of steel alloys.) Therefore, the ohmic losses in steel due to the magnets' presence would be higher and the braking time constant would become shorter (*i.e.*, the effective viscosity would become higher.)

- 1.f) In the laboratory flywheel experiment, if we define the output as a rotation angle and implement proportional control with unity negative feedback, can we achieve an arbitrary damping ratio?

Answer: In this case, the open-loop flywheel has one negative real pole and one integrator. Therefore, the root locus with proportional control looks as in the diagram below. For high enough gain, when the system becomes underdamped, the damping ratio is given by  $\zeta = \cos \theta$ . Therefore, by increasing the gain we can, in principle, achieve any damping ratio in the range  $(0, 1]$  (though, in practice the gain cannot increase indefinitely and, therefore, there is a lower limit in the values of damping ratio we can expect to achieve.) At low gains, the response is overdamped and the damping ratio can become arbitrarily high as the gain tends to zero.

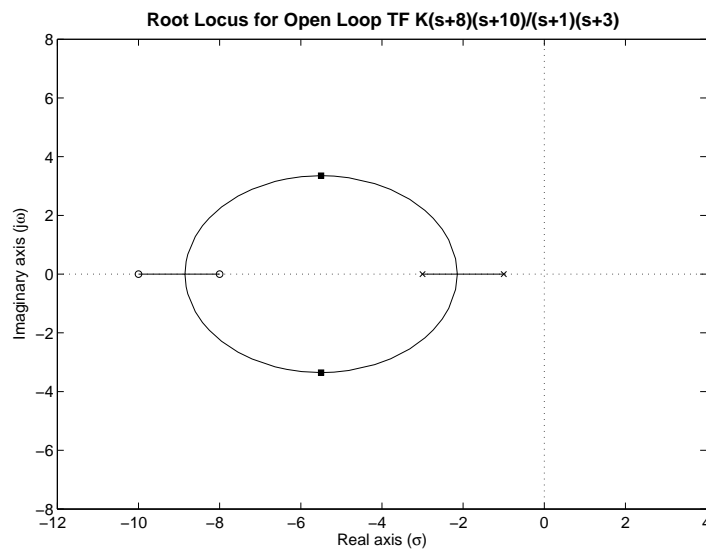
- 1.g) If an open-loop system has three or more negative real poles, no zeros, and we implement proportional control with unity negative feedback, is it safe to crank up the gain arbitrarily?

Answer: The Root Locus in this situation is as shown in the figure below. We can see that, for high enough gain, the closed-loop poles can enter the right-hand plane, *i.e.* the feedback system becomes unstable. That is not safe.



Problem 1, parts (f) and (g).

2. The root locus of a feedback system with open-loop poles at  $-1$ ,  $-3$  and open-loop zeros at  $-8$ ,  $-10$  is shown below. Study the diagram, then answer the following two questions:



- 2.a) Can the response of this system be tuned (approximately) to a settling time of 1sec with an appropriate choice of feedback gain  $K$ ? Justify your answer

graphically. If your answer is “yes,” there is no need to compute the value of  $K$  that would give this settling time.

Answer: Settling time of 1sec requires

$$T_s = \frac{4}{\zeta\omega_n} = 1 \Rightarrow \zeta\omega_n = 4$$

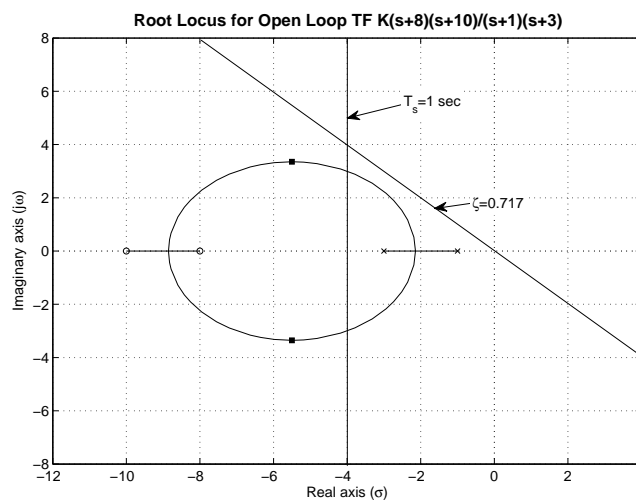
The given root locus does contain closed-loop poles whose real part equals  $-\zeta\omega_n = -4$  (see annotated root locus below;) therefore, it is possible to achieve settling time of 1sec by appropriate choice of feedback gain in the given system.

- 2.b)** Can the response of this system be tuned (approximately) to an overshoot of 4.32% with an appropriate choice of feedback gain  $K$ ? Justify your answer graphically. If your answer is “yes,” there is no need to compute the value of  $K$  that would give this amount of overshoot. Hint: A damping ratio of  $\zeta = 1/\sqrt{2}$  would approximately yield the desired overshoot value.

Answer: Damping ratio of  $\zeta = 1/\sqrt{2}$  requires that the closed-loop poles subtend angle  $\theta$  to the origin such that

$$\cos \theta = \zeta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ.$$

The given root locus does not intersect the line  $\theta = 45^\circ$  (see annotated root locus below;) therefore, there are no closed-loop poles for any value of feedback gain in the given system that yield the desired overshoot.



3. We are given a feedback system whose open-loop transfer function is

$$\frac{K(s + 20)}{(s + 2)(s + 4)(s + 10)},$$

where  $K$  is the feedback gain. In this problem, we will evaluate this system's closed-loop behavior using the root locus technique.

3.a) How many asymptotes are there in this system's root locus? What are the asymptote angles?

Answer: This system has  $\#p = 3$  finite poles and  $\#z = 1$  finite zero; therefore, it should have two open-loop zeros at infinity. The closed-loop poles should approach the open-loop zeros at infinity (as the feedback gain increases to infinity) along  $\#p - \#z = 2$  asymptotes at angles

$$\theta_a = \frac{(2m + 1)\pi}{3 - 1} = \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}.$$

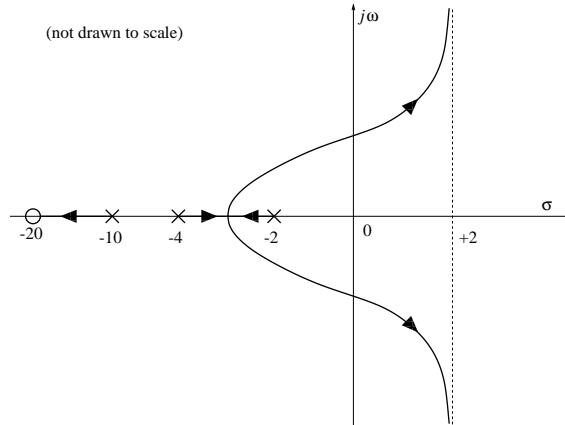
3.b) Where is the asymptotes' real-axis intercept?

Answer:

$$\sigma_a = \frac{\sum p_k - \sum z_k}{\#p - \#z} = \frac{-10 - 4 - 2 - (-20)}{3 - 1} = +2.$$

3.c) Sketch the root locus based on the information from the previous questions. There is no need to annotate break-in/away points or imaginary axis intercepts, if any.

Answer:

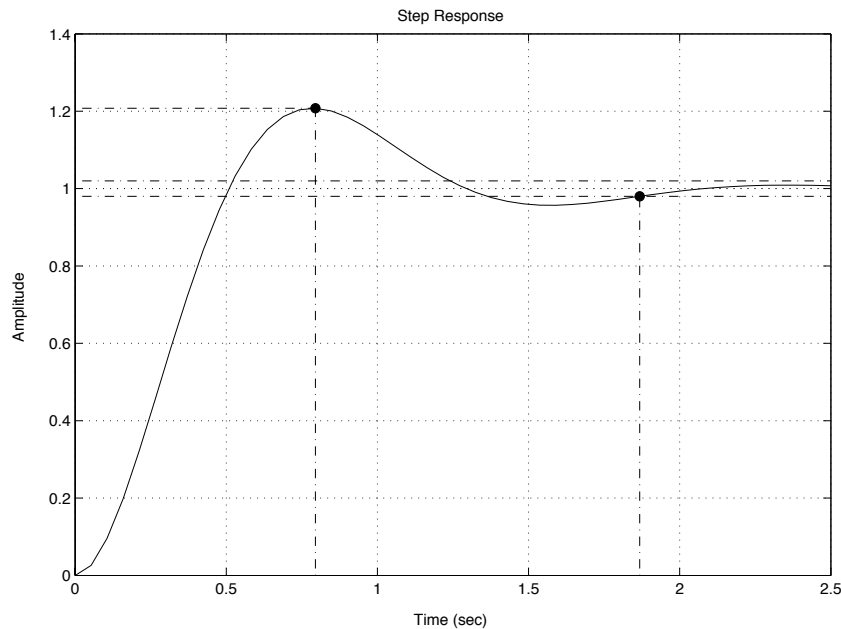


3.d) If you had to recommend this system to a customer, what would you advise with respect to increasing the feedback gain  $K$  indefinitely?

Answer: Since the root locus crosses over to the right-hand half-plane, the system will become unstable for a sufficiently high value of the feedback

gain. This is the most important warning one should give to a customer. In addition, large gain in the stable regime (just before the cross-over to instability) leads to increased overshoot and longer settling time; both of these qualities are generally undesirable.

4. The response of a proportional-control negative feedback system to a unit step-function input is shown below.



We are told that the open-loop transfer function of this system is 2<sup>nd</sup> order and contains no zeros.

- 4.a) Use the transient to provide two arguments in support of the assertion that this system is (at least) 2<sup>nd</sup> order.

Answer: The system's response is underdamped, with clear evidence of overshoot and oscillation; these phenomena do not occur in first-order systems. Therefore, it must be second order or higher. Moreover, at very small time  $t = 0^+$  we can see that the response is differentiable; this is also indicative of second order system behavior.

- 4.b) By further observing the transient, justify (or disprove) that the open-loop transfer function contains a free integrator.

Answer: The steady-state error appears to be zero, since the closed-loop response to the unit step-function approaches one at large times. That can only happen if there is a free integrator in the proportional control configuration.

- 4.c) Even closer observation of the above closed-loop transient yields the damped oscillation frequency  $\omega_d \approx 4\text{rad/sec}$  and settling time  $T_s \approx 2\text{sec}$ . Where is the second pole of the open-loop transfer function located?

Answer: For the second order system with proportional feedback, the Root Locus is as shown in the diagram below. From the observations, we deduce that the closed-loop (CL) poles are as indicated in the diagram, with

$$T_s = \frac{4}{\zeta\omega_n} = 2\text{sec} \Rightarrow \zeta\omega_n = 2.$$

Since the breakaway point is half-way between the two open-loop poles in this Root Locus, we deduce that the second open-loop pole is at  $s = -4$ .

- 4.d) What is the value of the feedback gain  $K$  that results in the closed-loop response shown above?

Answer: The closed-loop transfer function is

$$\frac{K}{s(s+4)} \cdot \frac{1}{1 + \frac{K}{s(s+4)}} = \frac{K}{s^2 + 4s + K}.$$

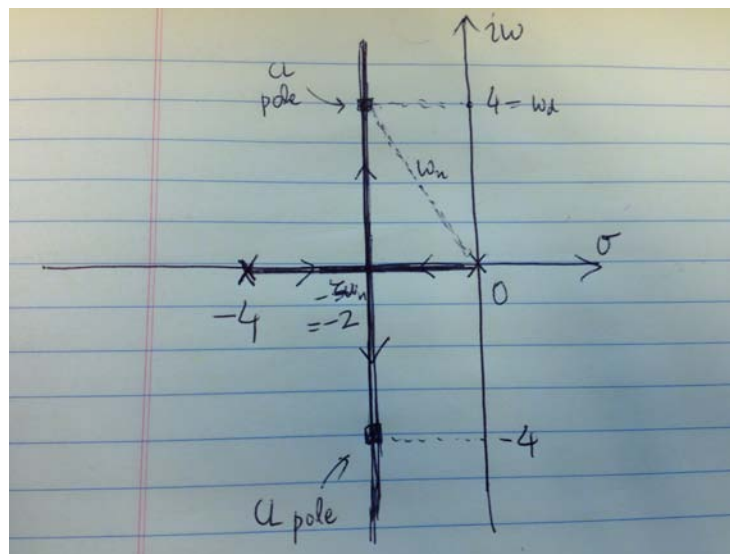
Comparing with the standard form for the second order system

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

we can see that  $K \equiv \omega_n^2$ . The natural frequency can be computed using Pythagoras theorem on a triangle with sides  $\omega_d$  and  $\zeta\omega_n$ , as shown, *i.e.*

$$\omega_n = \sqrt{4^2 + 2^2} = \sqrt{20}$$

and, hence,  $K = 20$ .



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