I-campus project

School-wide Program on Fluid Mechanics

Modules on Waves in fluids

T. R. Akylas & C. C. Mei

CHAPTER SIX

FORCED DISPERSIVE WAVES ALONG A NARROW CHANNEL

Linear surface gravity waves propagating along a narrow channel display interesting phenomena. At first we consider free waves propagating along an infinite narrow channel. We give the solution for this problem as a superposition of wave modes and we illustrate concepts like the notion of cut-off frequency. Second, we consider a semi-infinite channel with forced waves excited by a wave maker located at one end of the channel. As in the previous case, the wave field generated by the wave maker can be described as a superposition of wave modes. As the wave maker starts exciting the fluid, a wave front develop and starts propagating along the channel if the excitation frequency is above the cut-off frequency for the first channel wave mode. If the excitation frequency is below the cut-off frequency for the first channel mode, the wave disturbance stays localized close to the wave maker, and for the particular case where the excitation frequency matches the natural frequency of a particular channel wave modes, there is resonance between this particular wave mode and the wave maker, and the wave amplitude at the wave maker grows with time.

Effects of non-linearity and dissipation are not taken into account. In this chapter we obtain and illustrate through animations the free-surface displacement evolution in time along a semi-infinite narrow channel excited by a wave maker at one of its ends.

1 Free Wave Propagation Along a Narrow Waveguide.

We consider free waves propagating along an infinite channel of depth h and width $2b$. We adopt a coordinate system x, y, z, where x and z are in the horizontal plane and y is the vertical coordinate. The x axis is along the channel, the lateral walls are located at

$z = \pm b$ and the bottom is the plane $y = -h$. The free surface is located at $y = \eta(x, z, t)$, which is unknown. We assume irrotational flow and incompressible fluid such that the velocity field can be given as the gradient of a potential function $\phi(x, y, z, t)$, where t is the time parameterization. The linearized boundary value problem for propagation of free waves is given by the set of equations

$$\nabla^2\phi(x, y, z, t) = 0 \text{ for } -\infty < x < \infty, -h < y < 0 \text{ and } -b < z < b, \tag{1.1}$$

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0 \text{ at } y = 0, \tag{1.2}$$

$$\frac{\partial \phi}{\partial y} = 0 \text{ at } y = -h, \tag{1.3}$$

$$\frac{\partial \phi}{\partial z} = 0 \text{ at } z = \pm b, \tag{1.4}$$

and appropriate radiation conditions. This is an homogeneous boundary value problem that can be solved by the technique of separation of variables. First we assume that the free waves propagating along the channel are given as a superposition of plane monochromatic waves. Due to the linearity of the boundary value problem, we need only to solve it for a single nono-chromatic plane wave with wave frequency ω. The time dependence is

$$\exp(-i\omega t),$$

and now we can write the potential function $\phi(x, y, z, t)$ and the free-surface displacement $\eta(x, z, t)$ in the form

$$\phi(x, y, z, t) = \phi(x, y, z) \exp(-i\omega t), \tag{1.5}$$

$$\eta(x, z, t) = \eta(x, z) \exp(-i\omega t). \tag{1.6}$$

Now the boundary value problem given by equations (1.1) to (1.4) assume the form

$$\nabla^2\phi(x,y,z) = 0 \text{ for } -\infty < x < \infty, -h < y < 0 \text{ and } -b < z < b, \tag{1.7}$$

$$-\omega^2\phi + g\frac{\partial\phi}{\partial y} = 0 \text{ at } y = 0, \tag{1.8}$$

$$\frac{\partial\phi}{\partial y} = 0 \text{ at } y = -h, \tag{1.9}$$

$$\frac{\partial\phi}{\partial z} = 0 \text{ at } z = \pm b, \tag{1.10}$$

where we eliminated the free surface displacement $\eta(x,z)$ and reduced the boundary value problem to a boundary value problem in one dependent variable, $\phi(x,y,z)$. Next, we apply the technique of separation of variables to solve the boundary value problem given by equations (1.7) to (1.10). We assume the potential function $\phi(x,y,z)$ given as

$$\phi(x,y,z) \sim \exp(\pm ikx)\begin{pmatrix}\sin(k_z z)\\ \cos(k_z z)\end{pmatrix}H(y), \tag{1.11}$$

where the possible values k_z is determined by the boundary condition at the channel walls located at $z = \pm b$, and the possible values of the constant k are discussed below. If we substitute the expression given by equation (1.11) into the boundary value problem given by equations (1.7) to (1.10), we obtain a Sturm-Liouville problem (one-dimensional boundary value problem with a second order differential equation) for the function $H(y)$, which is given by the equations

$$H_{yy} + \Lambda H(y) = 0, \tag{1.12}$$

$$-\omega^2 H(y) + gH_y = 0 \text{ at } y = 0, \tag{1.13}$$

$$H_y = 0 \text{ at } y = -h, \tag{1.14}$$

where $\Lambda^2 = -k_z^2 + k^2$. The constant Λ represents a set of eigenvalues, which are functions of the wave frequency ω, of the gravity acceleration g and of the depth h.

If we apply the boundary conditions given by equation (1.10) to the potential function $\phi(x,y,z)$, we realize that we can use either $\cos(k_z z)$ or $\sin(k_z z)$ in the expression for $\phi(x,y,z)$ given by equation (1.11), but with different set of possible values for the constant k_z. The set of values for k_z are determined by the boundary condition (1.10) and the choice between $\cos(k_z z)$ and $\sin(k_z z)$. If we consider the z dependence of the potential $\phi(x, y, z)$ given in terms of $\cos(k_z z)$, the constant k_z has to assume the values

$$k_{zn} = \frac{\pi}{2b} \pm \frac{n\pi}{b} \text{ with } n \text{ as a natural number.} \tag{1.15}$$

If we consider the z dependence of the potential $\phi(x, y, z)$ given in terms of $\sin(k_z z)$, the constant k_z has to assume the values

$$k_{zm} = \pm\frac{m\pi}{b} \text{ with } m \text{ as a natural number.} \tag{1.16}$$

The general form of the solution for the equation (1.12) is

$$H(y) = A\cosh(\Lambda(y + h)) + B\sinh(\Lambda(y + h)), \tag{1.17}$$

but the boundary condition on the bottom given by the equation (1.14) implies that $B = 0$. The boundary condition at the free-surface ($y = 0$) gives the eigenvalue equation or *dispersion relation*

$$\omega^2 = g\Lambda\tanh(\Lambda h) \tag{1.18}$$

for the constant Λ. This implicit eigenvalue equation has one real solutions Λ_0 and an infinite countable set of pure imaginary eigenvalues $i\Lambda_l, l = 1, 2, \ldots$. Associated with these eigenvalues we have the eigenfunctions

$$H_0(y) = \frac{\cosh(\Lambda_0(y + h))}{\cosh(\Lambda_0 h)}, \tag{1.19}$$

$$H_l(y) = \frac{\cos(\Lambda_l(y + h))}{\cos(\Lambda_l h)}, \text{ with } l = 1, 2, \ldots \tag{1.20}$$

The term $\exp(ikx)(\exp(-ikx))$ in the equation (1.11) above for $\phi(x, y, z)$ represents a wave propagating to the right (left) if the constant k is real, or a right (left) evanescent wave if k is a pure imaginary number, or a combination of both if k is complex. We label the constant k as the *wavenumber*. Since, we are interested in free propagating waves, we need the constant k to be a real number. The value of this constant is given in terms of the constants Λ and k_z, according to the equation

$$k^2 = \Lambda^2 - k_z^2, \tag{1.21}$$

where the possible values of k_z are given by the equations (1.15) and (1.16). The possible values of Λ are solutions of the *dispersion relation* given by the equation (1.18). Since we want k as a real number, this excludes the imaginary solutions of the equation (1.18), so we can write the equation above in the form

$$k_n = \Lambda_0^2 - k_{zn}^2, \tag{1.22}$$

$$k_m = \Lambda_0^2 - k_{zm}^2, \tag{1.23}$$

where we appended the indexes n and m to the constant k to make clear its dependence on the eigenvalues k_{zn} and k_{zm}.

Now we can write the potential function $\phi(x, y, z)$ in the form

$$\begin{aligned}\phi(x, y, z) = &\sum_{m=-\infty}^{+\infty} \left\{ [A_m \exp(ik_m x) + B_m \exp(-ik_m x)] \frac{\cosh(\Lambda_0(y+h))}{\cosh(\Lambda_0 h)} \sin(k_{zm} z) \right\} \\ &+ \sum_{n=-\infty}^{\infty} \left\{ [A_n \exp(ik_n x) + B_n \exp(-ik_n x)] \frac{\cosh(\Lambda_0(y+h))}{\cosh(\Lambda_0 h)} \cos(k_{zn} z) \right\},\end{aligned} \tag{1.24}$$

and the free-surface displacement $\eta(x, z)$ is given by the equation

$$\begin{aligned}\eta(x, z) = -\frac{i\omega}{g} &\left\{ \sum_{m=-\infty}^{+\infty} \left[(A_m \exp(ik_m x) + B_m \exp(-ik_m x)) \frac{\cosh(\Lambda_0(y+h))}{\cosh(\Lambda_0 h)} \sin(k_{zm} z) \right] \right. \\ &\left. + \sum_{n=-\infty}^{\infty} \left[(A_n \exp(ik_n x) + B_n \exp(-ik_n x)) \frac{\cosh(\Lambda_0(y+h))}{\cosh(\Lambda_0 h)} \cos(k_{zn} z) \right] \right\},\end{aligned} \tag{1.25}$$

where the value of the constants A_m, A_n, B_m and B_n are specified by the appropriate radiation conditions.

According to the value of k_{zm} or k_{zn}, the constants k_m and k_n in the equations (1.24) and (1.25) may be real (propagating wave mode) or pure imaginary numbers (evanescent wave mode). If we fix the value of k_{zm} or k_{zn} (fix the value of m or n), for a given depth h, we can vary the wave frequency ω such that $\Lambda_0 > k_{zm}(k_{zn})$ or $\Lambda_0 < k_{zm}(k_{zn})$. When $\Lambda_0 > k_{zm}(k_{zn})$, $k_m(k_n)$ is a real number and we have a propagating wave mode, but when $\Lambda_0 < k_{zm}(k_{zn})$ we have that $k_m(k_n)$ is a pure imaginary number and the wave mode associated with this value of k is evanescent. So, the wave frequency value where $k_{zm} = \Lambda_0(k_{zn} = \Lambda_0)$ is called the *cut-off frequency* for the mth (nth) wave mode.

Next, we plot the dispersion relation given by equation (1.18) as a function of the wavenumber k and the depth h for various values of the eigenvalues k_{zm} (sine wave modes in the z coordinate) in the figures 1 and 2. As the value of k_{zm} increases (value of m increases), the wave frequency assume larger values for the considered range of the wavenumber k. The wave frequency value at $k = 0$ for a given k_{zm} (given m) is the *cut-off frequency* for the wave mode associated with the eigenvalue k_{zm}. For a fixed value of k_{zm}, frequencies below the *cut-off frequency* implies in pure imaginary wave numbers and the associated wave mode is exponentially decreasing (evanescent) or exponentially growing. Wave modes associated with pure imaginary wave numbers do not participate in the superposition leading to free waves solutions. According to figures 1 and 2, the higher the wave frequency, the higher the number of wave modes participating in the superposition leading to free waves solutions.

Another way to see that the wave modes associated with imaginary wave numbers (wave below the wave mode *cut-off frequency*) do not propagate is through the wave mode group velocity. In figures 3 and 4, we plot the group velocity for the first 10 wave modes associated with the eigenvalues k_{zm} (m from 0 to 9). For wave frequencies above the *cut-off frequency*, the considered wave mode (fixed value of k_{zm}) has a real wavenumber k and non-zero group velocity, as we can see through figures 3 and 4. As the wave frequency approaches the *cut-off frequency*, the group velocity of the considered wave mode approaches zero, according to figures 3 and 4. At the *cut-off frequency* of the considered wave mode, its group velocity is zero and no energy is transported by

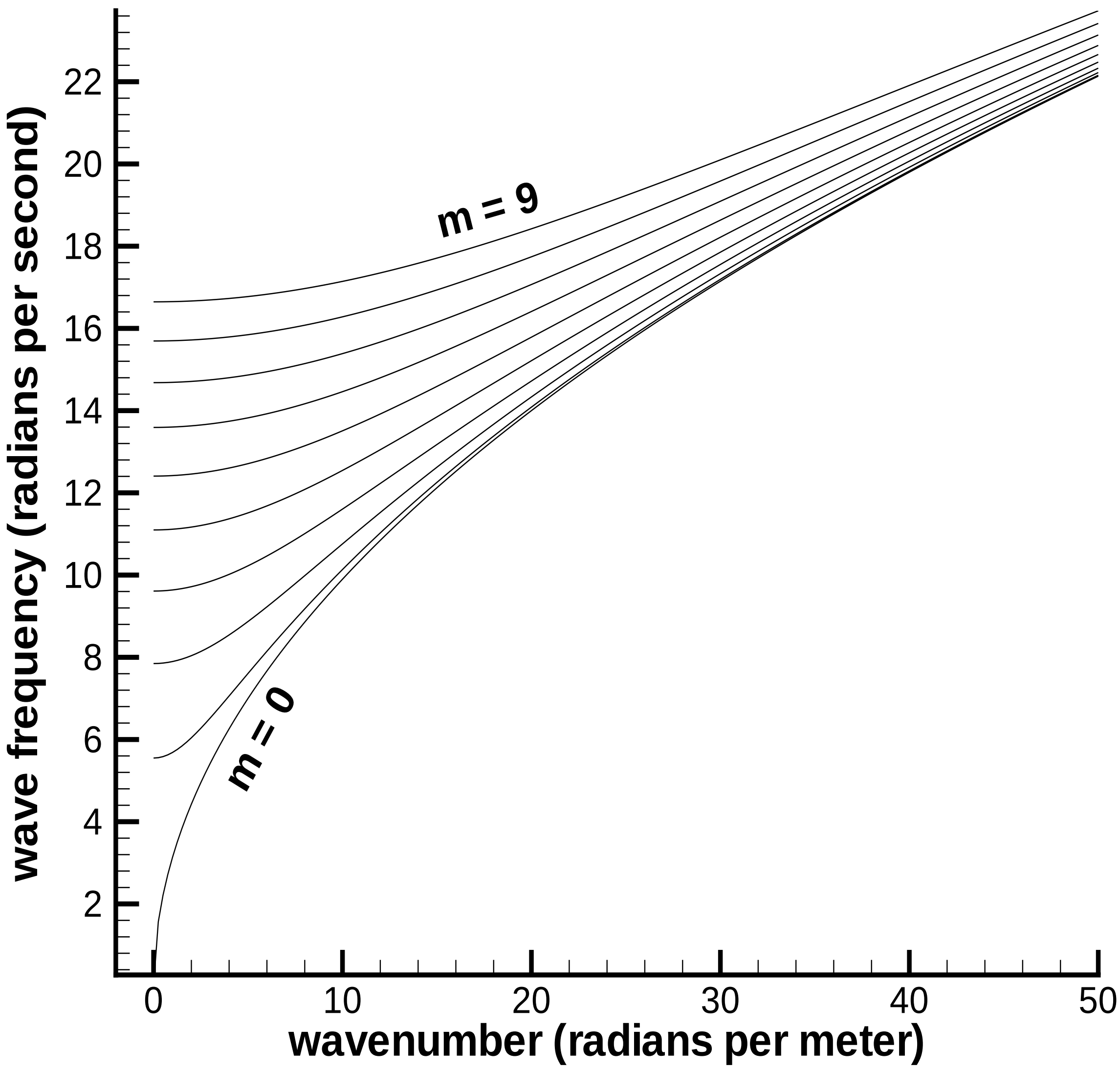

Figure 1: Wave frequency as a function of the wavenumber k for various values of the eigenvalue k_{zm} and water depth $h = 100$ meters.

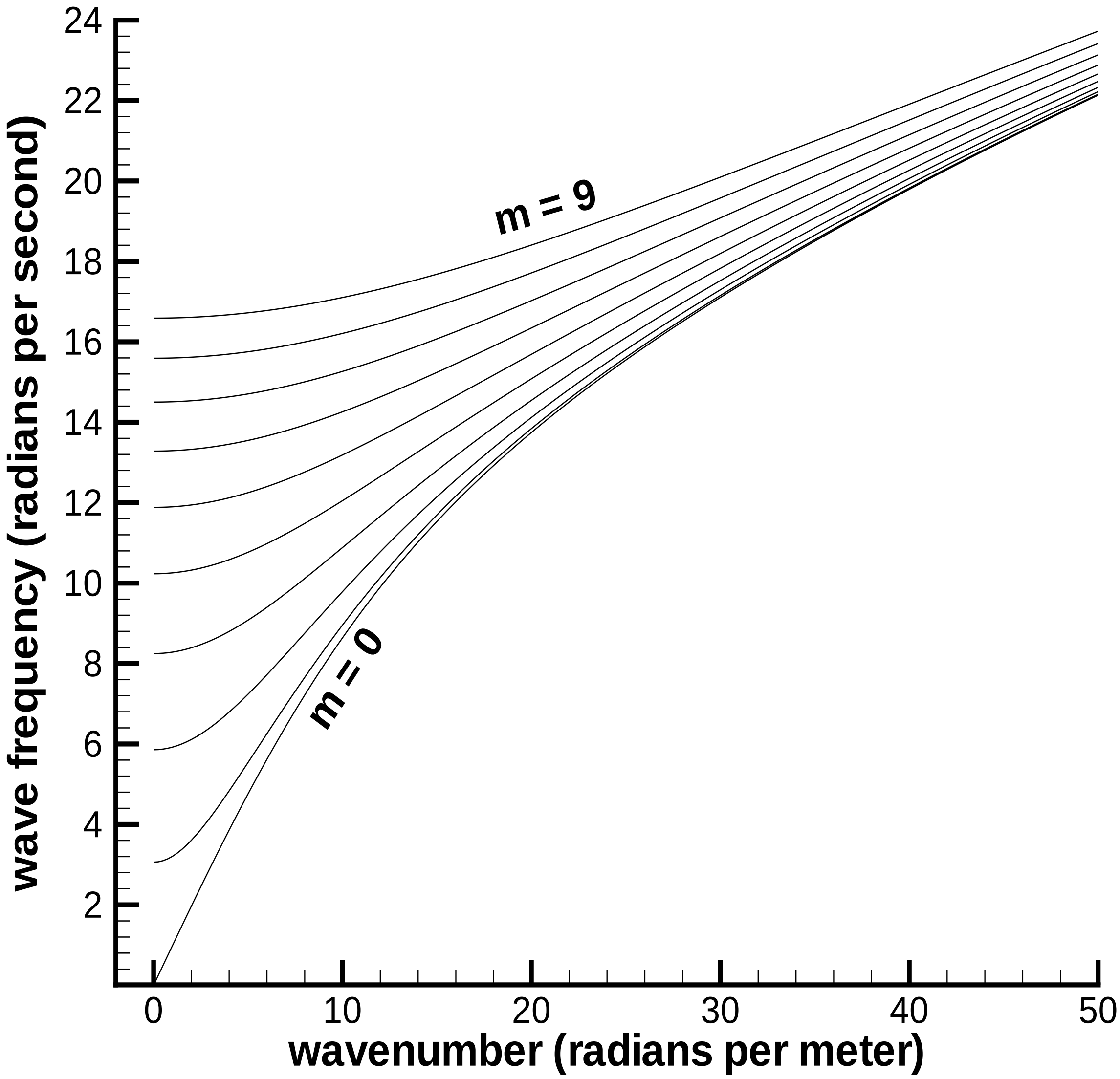

Figure 2: Wave frequency as a function of the wavenumber k for various values of the eigenvalue k_{zm} and water depth $h = 0.1$ meters.

this wave mode for wave frequencies at or below the wave mode *cut-off frequency.*

According to figures 3 and 4, the group velocity for each wave mode has a maximum value, which decays as the value of k_{zm} increases (value of m increases). The first wave mode (sine wave mode with $k_z = 0$) has the largest maximum group velocity, and since its *cut-off frequency* is zero, we can have free propagating waves for any wave frequency for the channel specified by its depth h, its width $2b$ and the gravity acceleration g. Above, we looked at the wave modes with sine dependence in the z coordinate. For the wave modes with cosine dependence in the z coordinate, the minimum absolute value of the eigenvalue k_{zn} is larger than the minimum absolute value for the eigenvalues k_{zm}, which is zero. Therefore, for any wave frequency we have free waves propagating along the channel. For the cosine wave modes there is a minimum *cut-off frequency.* Propagation of this type of wave mode is possible only for wave frequencies above their minimum *cut-off frequency.*

2 Forced Wave Propagation Along a Narrow Waveguide.

Now we consider forced waves propagating along a semi-infinite channel with the same depth h and width $2b$ as the channel in the previous section. The semi-infinite channel has a wave maker at one edge of the channel, which generates wave disturbances that may or may not propagate along the channel. The solution for the forced waves is given as a superposition of wave modes. The same wave modes we obtained in the previous section. Evanescent wave modes are also part of the solution in this case. They stay localized close to the wave maker and describe the local wave field. For a mono-chromatic excitation, the wave modes with *cut-off frequency* below the excitation frequency constitute the propagating wave field, and the wave modes with *cut-off frequency* above the excitation frequency are evanescent and stay localized close to the wave maker. Their superposition gives the evanescent wave field.

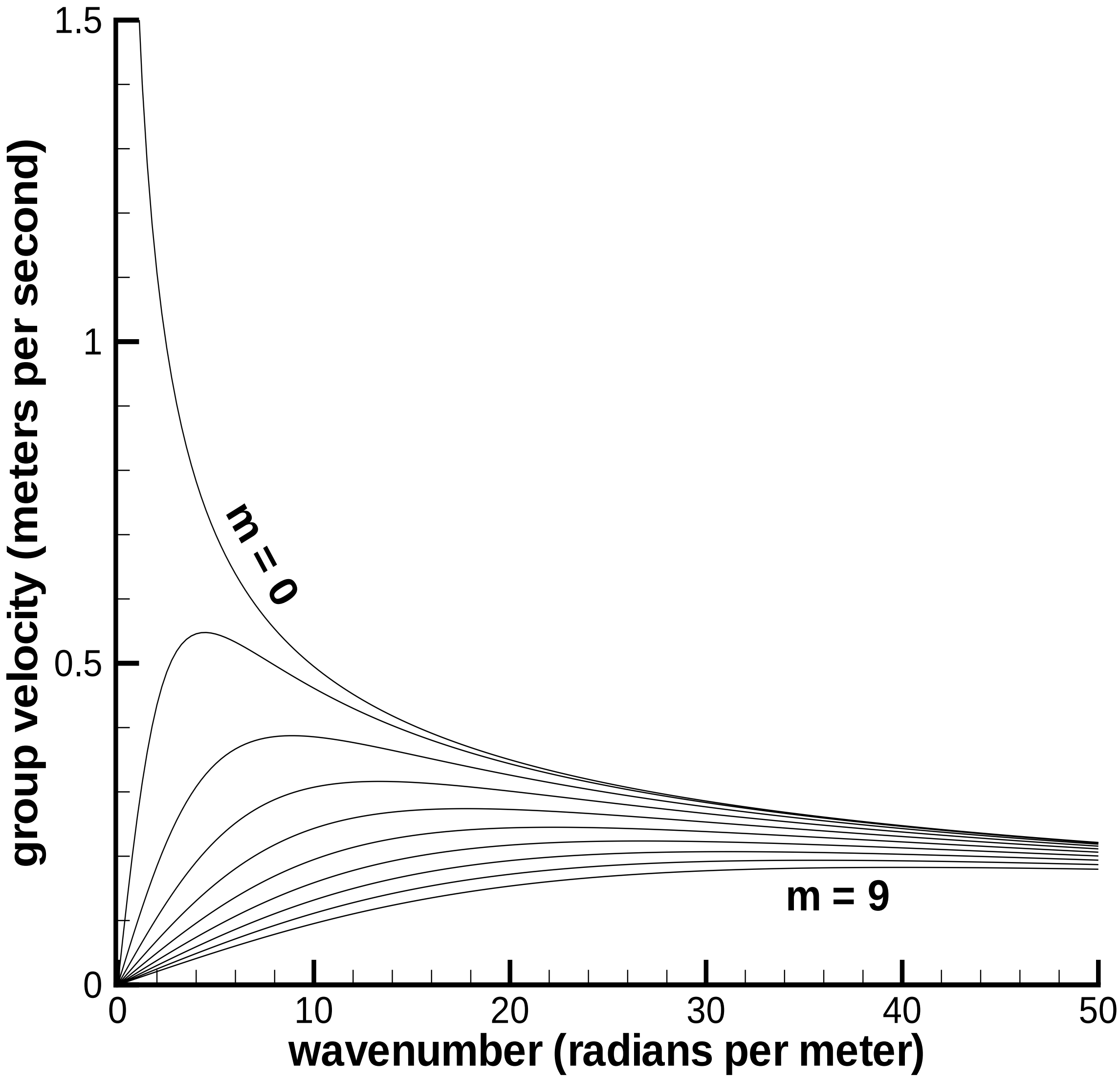

Figure 3: Group velocity as a function of the wavenumber k for various values of the eigenvalue k_{zm} and water depth $h = 100$ meters.

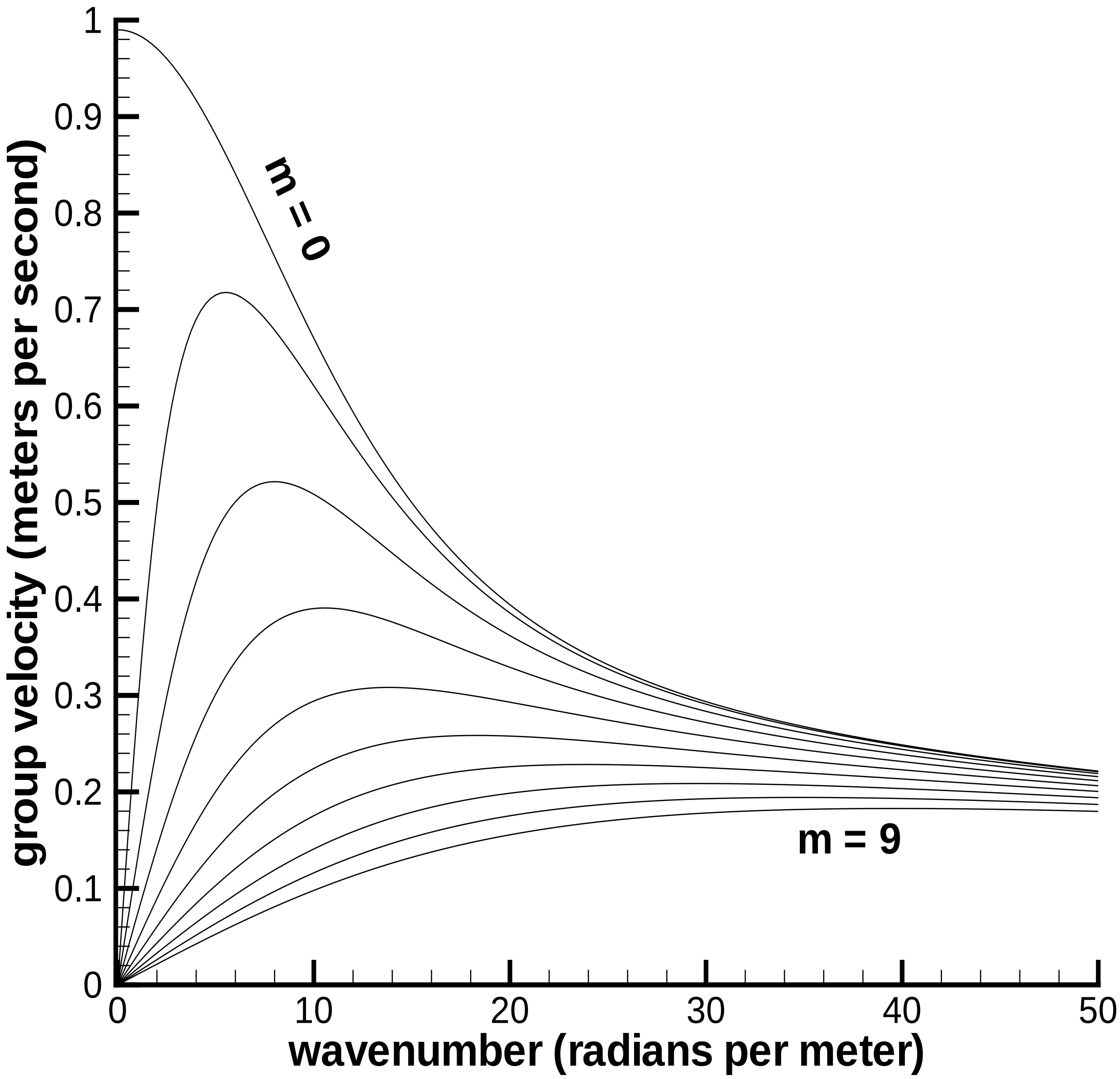

Figure 4: Group velocity as a function of the wavenumber k for various values of the eigenvalue k_{zm} and water depth $h = 0.1$ meters.

3 Initial Boundary Value Problem.

We consider the same coordinate system used in the previous section. The wave maker is located at $x = 0$ and the channel lays at $x > 0$. The linearized boundary value problem for the forced waves is similar to the boundary value problem for the free waves problem. The difference is the boundary condition describing the effect of the wave maker and the fact that the channel is now semi-infinite. The linear boundary value problem for forced waves is given by the set of equations

$$\nabla^2\phi(x,y,z,t) = 0 \text{ for } 0 < x < \infty, -h < y < 0 \text{ and } -b < z < b, \tag{3.26}$$

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0 \text{ at } y = 0, \tag{3.27}$$

$$\frac{\partial\phi}{\partial y} = 0 \text{ at } y = -h, \tag{3.28}$$

$$\frac{\partial\phi}{\partial z} = 0 \text{ at } z = \pm b, \tag{3.29}$$

$$\frac{\partial\phi}{\partial x} = \frac{\omega A}{b}F(z)G(y)f(t) \text{ on } x = 0, \tag{3.30}$$

and the free surface displacement $\eta(x,z,t)$ is related to the potential function $\phi(x,y,z,t)$ according to the equation

$$\eta(x,z,t) = -\frac{1}{g}\frac{\partial\phi}{\partial t}(x,0,z,t). \tag{3.31}$$

The function $f(t)$ is a known function of time. Actually, we chose an harmonic excitation, so we have

$$f(t) = \cos(\omega t), \tag{3.32}$$

where ω is the excitation frequency. We need also to consider initial conditions for the boundary value problem above. They are given by the equations

$$\phi(x,y,z,0) = 0, \tag{3.33}$$

$$\phi_t(x,y,z,0) = 0, \tag{3.34}$$

where the initial condition (3.34) is equivalent to have a still free surface at $t = 0$ ($\eta(x, z, 0) = 0$). Next, we solve the initial boundary value problem, which is discussed in the next section.

3.1 Solution of the Initial Boundary Value Problem.

The first step to solve the initial boundary value problem given by equations (3.26) to (3.30) is to apply the cosine transform in the x variable. This results in a non-homogeneous Helmholtz-like equation for the potential function under homogeneous boundary conditions. Since the resulting equation is non-homogeneous, the solution is given as the superposition of the solution for the homogeneous part of the problem plus a particular solution that handles the non-homogeneity. To solve the associated homogeneous problem, we use the method of separation of variables as in the previous section. The solution of the homogeneous problem is given as a superposition of modes in the y and z variables. The particular solution is obtained using the homogeneous solution through the method of variation of the parameters. The constants of the homogeneous solution are obtained by applying the boundary conditions to the full solution (homogeneous plus particular solutions). Next, we discuss in detail the steps outlined above.

We consider the cosine transform pair

$$\hat{f}(k) = \int_0^\infty f(x)\cos(kx)dx \tag{3.35}$$

and

$$f(x) = \frac{1}{2\pi}\int_0^\infty \hat{f}(k)\cos(kx)dk. \tag{3.36}$$

If we apply the cosine transform (3.36) to the second partial derivative of the potential function $\phi(x, y, z, t)$ with respect to the x variable, we have that

$$\int_0^\infty \phi_{xx}\cos(kx)dx = -\phi_x(0, y, z, t) - k^2\hat{\phi}(k, y, z, t), \tag{3.37}$$

since we assumed that $\phi_x \to 0$ and $\phi \to 0$ as $x \to \infty$. The term $\phi_x(0, y, z, t)$ is specified by the boundary condition at $x = 0$ and given by equation (3.30). Next, we apply the cosine transform to the initial boundary value problem given by equations (3.26) to (3.30). This results in the set of equations

$$\hat{\phi}_{yy} + \hat{\phi}_{zz} - k^2\hat{\phi} = \phi_x(0, y, z, t) = \frac{A\omega}{b} F(z)G(y)\cos(\omega t), \tag{3.38}$$

$$\hat{\phi}_{tt} + g\phi_y = 0 \text{ on } y = 0, \tag{3.39}$$

$$\hat{\phi}_y = 0 \text{ on } y = -h, \tag{3.40}$$

$$\hat{\phi}_z = 0 \text{ on } z = \pm b, \tag{3.41}$$

with the initial conditions given by equations (3.33) and (3.34) written in the form

$$\hat{\phi}(k, y, z, 0) = 0, \tag{3.42}$$

$$\hat{\phi}_t(k, y, z, 0) = 0. \tag{3.43}$$

This is a non-homogeneous initial boundary value problem for the function $\hat{\phi}(k, y, z, t)$ (cosine transform of $\phi(x, y, z, t)$). Our strategy to solve this initial boundary value problem is to find the general form of the solution of the homogeneous part of the initial boundary value problem given by equations (3.38) to (3.41) plus a particular solution for the non-homogeneous part of this initial boundary value problem. To find the value of the constants of the homogeneous part of the solution, we apply the initial and boundary conditions to the full solution (homogeneous plus particular). Next, we consider the homogeneous part of the initial boundary value problem for $\hat{\phi}$, which is given as the superposition of wave modes obtained in the previous section. So, the solution of the homogeneous problem is similar to the one given by equation (1.24). The solution for the homogeneous problem is

$$\hat{\phi}_H = \sum_{n=-\infty}^{+\infty} \{[A_n(k,t)\cosh(\Lambda_n(y+h)) + B_n(k,t)\sinh(\Lambda_n(y+h))]\cos(k_{zn}z)\}$$
$$+ \sum_{m=-\infty}^{+\infty} \{[C_m(k,t)\cosh(\Lambda_m(y+h)) + D_m(k,t)\sinh(\Lambda_m(y+h))]\sin(k_{zm}z)\},$$

where $\Lambda_m^2 = k^2 + k_{zm}^2, \Lambda_n^2 = k^2 + k_{zn}^2$, and k_{zn} and k_{zm} are given respectively, in equations (1.16) and (1.15). As we mentioned before, the general solution is given as a superposition of the homogeneous solution $\hat{\phi}_H$ plus a particular solution. We assume that the particular solution has the form

$$\begin{aligned}\hat{\phi}_P = &\sum_{n=-\infty}^{+\infty} \left\{ \left[\tilde{A}_n(k,y,t)\cosh(\Lambda_n(y+h)) + \tilde{B}_n(k,y,t)\sinh(\Lambda_n(y+h)) \right] \cos(k_{zn}z) \right\} \\ &+ \sum_{m=-\infty}^{+\infty} \left\{ \left[\tilde{C}_m(k,y,t)\cosh(\Lambda_m(y+h)) + \tilde{D}_m(k,y,t)\sinh(\Lambda_m(y+h)) \right] \sin(k_{zm}z) \right\}.\end{aligned}$$

We substitute the potential $\hat{\phi}_P$ in the non-homogeneous Helmholtz equation (3.38) in the y and z variables. We also impose that

$$\begin{aligned}\frac{\partial \hat{\phi}_P}{\partial y} = &\sum_{n=-\infty}^{+\infty} \left\{ \Lambda_n \left[\tilde{A}_n \sinh(\Lambda_n(y+h)) + \tilde{B}_n \cosh(\Lambda_n(y+h)) \right] \cos(k_{zn}z) \right\} \\ &+ \sum_{m=-\infty}^{+\infty} \left\{ \Lambda_m \left[\tilde{C}_m \sinh(\Lambda_m(y+h)) + \tilde{D}_m \cosh(\Lambda_m(y+h)) \right] \sin(k_{zm}z) \right\}.\end{aligned} \tag{3.44}$$

The procedure above results in the set of equations for the amplitudes $\tilde{A}_n, \tilde{B}_n, \tilde{C}_m$ and $\tilde{D}_m$.

$$(\tilde{A}_n)_y \cosh(\Lambda_n(y+h)) + (\tilde{B}_n)_y \sinh(\Lambda_n(y+h)) = 0, \tag{3.45}$$

$$(\tilde{C}_m)_y \cosh(\Lambda_m(y+h)) + (\tilde{D}_m)_y \sinh(\Lambda_m(y+h)) = 0, \tag{3.46}$$

$$\Lambda_n \left\{ (\tilde{A}_n)_y \sinh(\Lambda_n(y+h)) + (\tilde{B}_n)_y \cosh(\Lambda_n(y+h)) \right\} = \frac{A\omega}{b^2} G(y)\cos(\omega t) F_n, \tag{3.47}$$

$$\Lambda_m \left\{ (\tilde{C}_m)_y \sinh(\Lambda_m(y+h)) + (\tilde{D}_m)_y \cosh(\Lambda_m(y+h)) \right\} = \frac{A\omega}{b^2} G(y)\cos(\omega t) F_m, \tag{3.48}$$

where

$$F_m = \int_{-b}^{b} F(z)\sin(k_{zm}z)dz, \tag{3.49}$$

$$F_n = \int_{-b}^{b} F(z)\cos(k_{zn}z)dz. \tag{3.50}$$

If we solve the set of equations above and integrate with respect to the y variable from $-h$ to 0, we obtain the following expressions for the amplitudes $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n$ and $\tilde{D}_n$, which follows:

$$\tilde{A}_n = -\frac{A\omega}{b^2\Lambda_n}\cos(\omega t)F_n G_n(y), \tag{3.51}$$
$$\tilde{B}_n = \frac{A\omega}{b^2\Lambda_n}\cos(\omega t)F_n H_n(y), \tag{3.52}$$
$$\tilde{C}_m = -\frac{A\omega}{b^2\Lambda_m}\cos(\omega t)F_m G_m(y), \tag{3.53}$$
$$\tilde{D}_m = \frac{A\omega}{b^2\Lambda_m}\cos(\omega t)F_m H_m(y), \tag{3.54}$$

where the functions $G_n(y), H_n(y), G_m(y)$ and $H_m(y)$ are given by the equations

$$G_n(y) = \int_{-h}^{y} G(p)\sinh(\Lambda_n(p+h))dp, \tag{3.55}$$
$$H_n(y) = \int_{-h}^{y} G(p)\cosh(\Lambda_n(p+h))dp, \tag{3.56}$$
$$G_m(y) = \int_{-h}^{y} G(p)\sinh(\Lambda_m(p+h))dp, \tag{3.57}$$
$$H_m(y) = \int_{-h}^{y} G(p)\cosh(\Lambda_m(p+h))dp. \tag{3.58}$$

Now, the total solution $\hat{\phi}(k, y, z)$ can written in the form

$$\begin{aligned}\hat{\phi} = &\sum_{n=-\infty}^{\infty}\left\{\left[\left(A_n - \frac{A\omega}{b^2\Lambda_n}\cos(\omega t)F_n G_n(y)\right)\cosh(\Lambda_n(y+h))\right.\right.\\ &\left.\left.+\left(B_n + \frac{A\omega}{b^2\Lambda_n}\cos(\omega t)F_n H_n(y)\right)\sinh(\Lambda_n(y+h))\right]\cos(k_{zn}z)\right\}\\ &+\sum_{m=-\infty}^{\infty}\left\{\left[\left(C_m - \frac{A\omega}{b^2\Lambda_m}\cos(\omega t)F_m G_m(y)\right)\cosh(\Lambda_m(y+h))\right.\right.\\ &\left.\left.+\left(D_m + \frac{A\omega}{b^2\Lambda_m}\cos(\omega t)F_m H_m(y)\right)\sinh(\Lambda_m(y+h))\right]\sin(k_{zm}z)\right\}.\end{aligned} \tag{3.59}$$

In the expression above we still need to obtain the constants A_m, B_m, C_n and D_n of the homogeneous part of the solution. To do so, we apply the boundary conditions (3.39) at $y = 0$ and (3.40) at $y = -h$. The boundary condition at $y = -h$, given by the equation (3.40), implies that $D_m = 0 (B_n = 0)$. The boundary condition at $y = 0$ gives the equation

$$
\begin{aligned}
(A_n)_{tt} + g\Lambda_n \tanh(\Lambda_n h) A_n = & \frac{A}{b^2}\frac{F_n}{\Lambda_n}\left\{\omega^3 \cos(\omega t)\left[H_n(0)\tanh(\Lambda_n h) - G_n(0)\right]\right. \\
& \left. + g\Lambda_n \omega \cos(\omega t)\left[G_n(0)\tanh(\Lambda_n h) - H_n(0)\right]\right\}.
\end{aligned}
\tag{3.60}
$$

We also obtain a similar equation for C_m. This is a non-homogeneous second order differential equation in time for the amplitude A_n. Its solution is given as the superposition of the solution of the homogeneous part of the equation plus a particular solution which satisfies the non-homogeneous term in the equation (3.60). The homogeneous solution is given as

$$
(A_n(t))_H = \hat{A}\cos(\Omega_n t) + \hat{B}\sin(\Omega_n t) \tag{3.61}
$$

with $\Omega_n^2 = g\Lambda_n \tanh(\Lambda_n h)$. We assume the particular solution given in the form

$$
(A_n(t))_P = \hat{A}(t)_P \cos(\Omega t) + \hat{B}(t)_P \sin(\Omega t). \tag{3.62}
$$

We impose that

$$
\frac{d}{dt}(A_n(t))_P = \Omega_n \left\{-\hat{A}(t)_P \sin(\Omega t) + \hat{B}(t)_P \cos(\Omega t)\right\}. \tag{3.63}
$$

If we substitute the form of the particular solution, given by equation (3.62) into the governing equation (3.61) and take into account the assumed form for $\frac{d}{dt}(A_n(t))_P$, given by equation (3.63), we obtain for the amplitudes $\hat{A}(t)_P$ and $\hat{B}(t)_P$ the expressions

$$
\hat{A}(t)_P = \frac{1}{2}\frac{\psi(\omega,\Omega_n,h)}{\Omega_n}\left\{\frac{\cos[(\Omega_n-\omega)t]}{\Omega_n-\omega} + \frac{\cos[(\Omega_n+\omega)t]}{\Omega_n+\omega}\right\}, \tag{3.64}
$$

$$
\hat{B}(t)_P = \frac{1}{2}\frac{\psi(\omega,\Omega_n,h)}{\Omega_n}\left\{\frac{\sin[(\Omega_n-\omega)t]}{\Omega_n-\omega} + \frac{\sin[(\Omega_n+\omega)t]}{\Omega_n+\omega}\right\}, \tag{3.65}
$$

where

$$\begin{aligned}\psi(\omega,\Omega_n,h) =& \frac{A}{b^2}\frac{F_n}{\Lambda_n}\left\{\omega^3\left[H_n(0)\tanh(\Lambda_n h) - G_n(0)\right]\right. \\ &\left. +g\Lambda_n\omega\left[G_n(0)\tanh(\Lambda_n h) - H_n(0)\right]\right\}\end{aligned} \tag{3.66}$$

If we substitute these expressions for the amplitudes $\hat{A}(t)_P$ and $\hat{B}(t)_P$ in the assumed form of the particular solution, we obtain

$$(A_n(t))_P = -\frac{\psi(\omega,\Omega_n,h)}{\omega^2-\Omega_n^2}\cos(\omega t). \tag{3.67}$$

As a result, we obtain for $A_n(t)$ the following expression:

$$A_n(t) = \hat{A}_n\cos(\Omega_n t) + \hat{B}_n\sin(\Omega_n t) - \frac{\psi(\omega,\Lambda,h)}{(\omega^2-\Omega^2)}\cos(\omega t) \tag{3.68}$$

For the amplitude C_m we obtain the same expression as above for $A_n(t)$, but with the index m instead of the index n. Now the potential function can be written in the form

$$\begin{aligned}\hat{\phi} = &\sum_{n=-\infty}^{+\infty}\left\{\left[\left(\hat{A}_n\cos(\Omega_n t) + \hat{B}_n\sin(\Omega_n t) - \frac{\psi(\omega,\Lambda_n,h)}{(\omega^2-\Omega^2)}\cos(\omega t)\right.\right.\right. \\ &\left.\left.\left.-\frac{A}{b^2}\omega\cos(\omega t)\frac{F_n}{\Lambda_n}G_n(y)\right)\cosh(\Lambda_n(y+h)) + \frac{A}{b^2}\omega\cos(\omega t)\frac{F_n}{\Lambda_n}H_n(y)\sinh(\Lambda_n(y+h))\right]\cos(k_{zn}z)\right\} \\ &+\sum_{m=-\infty}^{+\infty}\left\{\left[\left(\hat{C}_m\cos(\Omega_m t) + \hat{D}_m\sin(\Omega_m t) - \frac{\psi(\omega,\Lambda_m,h)}{(\omega^2-\Omega_m^2)}\cos(\omega t)\right.\right.\right. \\ &\left.\left.\left.-\frac{A}{b^2}\omega\cos(\omega t)\frac{F_n}{\Lambda_m}G_m(y)\right)\cosh(\Lambda_m(y+h)) + \frac{A}{b^2}\omega\cos(\omega t)\frac{F_m}{\Lambda_m}H_m(y)\sinh(\Lambda_m(y+h))\right]\sin(k_{zm}z)\right\},\end{aligned} \tag{3.69}$$

which is a function of the unknown constants $\hat{A}_n, \hat{B}_n, \hat{C}_m$ and $\hat{D}_m$. To obtain these constants we use the initial conditions for $\hat{\phi}(k,y,z,t)$ given by equations (3.42) and (3.43). We obtain

$$\hat{A}_n = \frac{\psi(\omega, \Lambda_n, h)}{\omega^2 - \Omega_n^2} + \frac{A}{b^2}\frac{\omega F_n}{\Lambda_n} G_n(0) - \frac{A}{b^2}\frac{\omega F_n}{\Lambda_n} H_n(0)\tanh(\Lambda_n h), \tag{3.70}$$

$$\hat{B}_n = 0, \tag{3.71}$$

$$\hat{C}_m = \frac{\psi(\omega, \Lambda_m, h)}{\omega^2 - \Omega_m^2} + \frac{A}{b^2}\frac{\omega F_m}{\Lambda_m} G_m(0) - \frac{A}{b^2}\frac{\omega F_m}{\Lambda_m} H_m(0)\tanh(\Lambda_m h), \tag{3.72}$$

$$\hat{D}_m = 0. \tag{3.73}$$

The final form of the potential function $\hat{\phi}(k, y, z, t)$ is given by the equation

$$\begin{aligned}
\hat{\phi} = \sum_{n=-\infty}^{+\infty} \Bigg\{ \frac{AF_n}{b^2} \Bigg[&-\frac{\omega^3}{\lambda_n^2(\omega^2 - \Omega_n^2)} \frac{\cos(\Omega_n t) - \cos(\omega t)}{\cosh(\Lambda_n h)} + \omega \frac{\cosh(\Lambda_n h)}{\Lambda^2} \left(\cos(\Omega_n t) - \cos(\omega t)\right) \\
&- \frac{\omega \sinh^2(\Lambda_n h)}{\Lambda_n^2 \cosh(\Lambda_n h)} \cos(\Omega_n t) \Bigg] \cosh(\Lambda_n (y+h)) + \frac{A}{b^2} \omega F_n \frac{\sinh^2(\Lambda_n (y+h))}{\Lambda_n^2} \cos(\omega t) \Bigg\} \cos(k_{zn} z) \\
+ \sum_{m=-\infty}^{+\infty} \Bigg\{ \frac{AF_m}{b^2} \Bigg[&-\frac{\omega^3}{\lambda_m^2(\omega^2 - \Omega_m^2)} \frac{\cos(\Omega_m t) - \cos(\omega t)}{\cosh(\Lambda_m h)} + \omega \frac{\cosh(\Lambda_m h)}{\Lambda^2} \left(\cos(\Omega_m t) - \cos(\omega t)\right) \\
&- \frac{\omega \sinh^2(\Lambda_m h)}{\Lambda_m^2 \cosh(\Lambda_m h)} \cos(\Omega_m t) \Bigg] \cosh(\Lambda_m (y+h)) + \frac{A}{b^2} \omega F_m \frac{\sinh^2(\Lambda_m (y+h))}{\Lambda_m^2} \cos(\omega t) \Bigg\} \sin(k_{zm} z).
\end{aligned} \tag{3.74}$$

We are interested in the displacement of the free-surface $\eta(k, z, t)$, which is given in terms of the potential function $\phi(k, y, z, t)$ according to the equation (3.31). Then the cosine transform of the free-surface displacement is given in terms of the Fourier transform of the potential according to the equation

$$\hat{\eta}(k, z, t) = -\frac{1}{g}\frac{\partial \hat{\phi}}{\partial t}(x, 0, z, t). \tag{3.75}$$

If we apply this equation to the expression for $\hat{\phi}(k, y, z, t)$ given by equation (3.74), we obtain

$$\hat{\eta}(k,z,t) = \sum_{n=-\infty}^{\infty} \left\{ \left[\frac{AF_n}{gb^2} \frac{\omega\Omega_n}{\Lambda_n^2(\omega^2-\Omega_n^2)} \left(\omega\sin(\omega t) - \Omega_n \sin(\Omega_n t)\right) \right] \cos(k_{zn}z) \right\}$$
$$+ \sum_{m=-\infty}^{\infty} \left\{ \left[\frac{AF_m}{gb^2} \frac{\omega\Omega_m}{\Lambda_m^2(\omega^2-\Omega_m^2)} \left(\omega\sin(\omega t) - \Omega_m \sin(\Omega_m t)\right) \right] \cos(k_{zm}z) \right\}. \tag{3.76}$$

3.2 Fourier Integral Solution.

Here we apply the inverse cosine transform to the expression above for the cosine transform of the free-surface displacement. The inverse cosine transform is given by equation (3.36), and we apply it to the equation (3.76) to obtain the free-surface displacement

$$\eta(x,z,t) = \sum_{n=-\infty}^{\infty} \left\{ \frac{AF_n}{gb^2} \left[\frac{1}{2\pi} \int_0^{\infty} \frac{\omega\Omega_n}{\Lambda_n^2(\omega^2-\Omega_n^2)} \left(\omega\sin(\omega t) - \Omega_n \sin(\Omega_n t)\right) \cos(kx)dk \right] \cos(k_{zn}z) \right\}$$
$$+ \sum_{m=-\infty}^{\infty} \left\{ \frac{AF_m}{gb^2} \left[\frac{1}{2\pi} \int_0^{\infty} \frac{\omega\Omega_m}{\Lambda_m^2(\omega^2-\Omega_m^2)} \left(\omega\sin(\omega t) - \Omega_m \sin(\Omega_m t)\right) \cos(kx)dk \right] \sin(k_{zm}z) \right\} \tag{3.77}$$

The integrands in the integrals above apparently have poles in the complex k plane for wave numbers solutions of $\omega^2 - \Omega_n^2(k) = 0$. As $\Omega_n(k)$ approaches $\pm\omega$, we have that $\Omega_n(k)\sin(\omega t)$ approaches $\omega\sin(\omega t)$ in the same fashion, so there is no singularity in the integrand and the integral is well behaved. To obtain the free-surface displacement we evaluated numerically the inverse cosine transforms appearing in equation (3.77). Results from these simulations were used to generate animations of the evolution of the free-surface displacement due to the action of the wave maker over the fluid. These animations are discussed in the next section.

3.3 Numerical Results.

Here we show results from the numerical evaluation of the inverse cosine transforms appearing in the equation (3.77) for the free-surface displacement. We display the evolution of the free-surface displacement in time through the numerical evaluation of equation (3.77). We generated animations for the evolution of the free-surface displacement due to the action of the wave maker at $x = 0$. Here we discuss the examples and we give links for the movies associated with these examples.

- We consider that the displacement of the wave maker coincides with the first cosine wave mode in the z direction. The excitation frequency is above the *cut-off frequency* for the first cosine wave mode. With this type of excitation, the only wave mode taking part in the solution is the first cosine wave mode. Since the wave maker starts from rest to the harmonic motion, it excites initially all wave frequencies and generates a transient which propagates along the channel and is followed by a nono-chromatic wave train (the cosine wave mode) with frequency equals to the excitation frequency. The transient has a wave front which propagates with the maximum group velocity possible for this cosine wave mode. For the depth $h = 0.1$ meters, figure 5 illustrates the maximum group velocity for the cosine wave modes. The maximum group velocity possible $C_{g,max}$ is the group velocity of the cosine wave mode with $k_{zn} = \frac{\pi}{2b}$ $(n = 0)$. Then, for a given time instant t, there is no wave disturbance at positions $x > C_{g,max}t$. The transient for a given instant t stays in the region $C_{g,max}t > x > C_g(\omega)t$, where $C_g(\omega)$ is the group velocity of the excited cosine wave mode at the excitation frequency ω. To see the animation associated with this example, click here.

- We consider that the displacement of the wave maker coincides with the second cosine wave mode in the z direction. The excitation frequency is above the *cut-off frequency* for the first cosine mode but below the *cut-off frequency* for the second cosine mode. Again, the wave maker starts from rest to the harmonic motion. All wave frequencies are excited initially and a transient develops. The transient propagates along the channel, and behind it we are left with only the second cosine wave mode, which decays exponentially as we go away from the wave maker, since at this excitation frequency the second cosine wave mode is evanescent. Again, the transient has a wave front which propagates with the maximum group velocity possible for the second cosine wave mode. To see the animation associated with this example, click here.

- We consider that the displacement of the wave maker coincides with the first

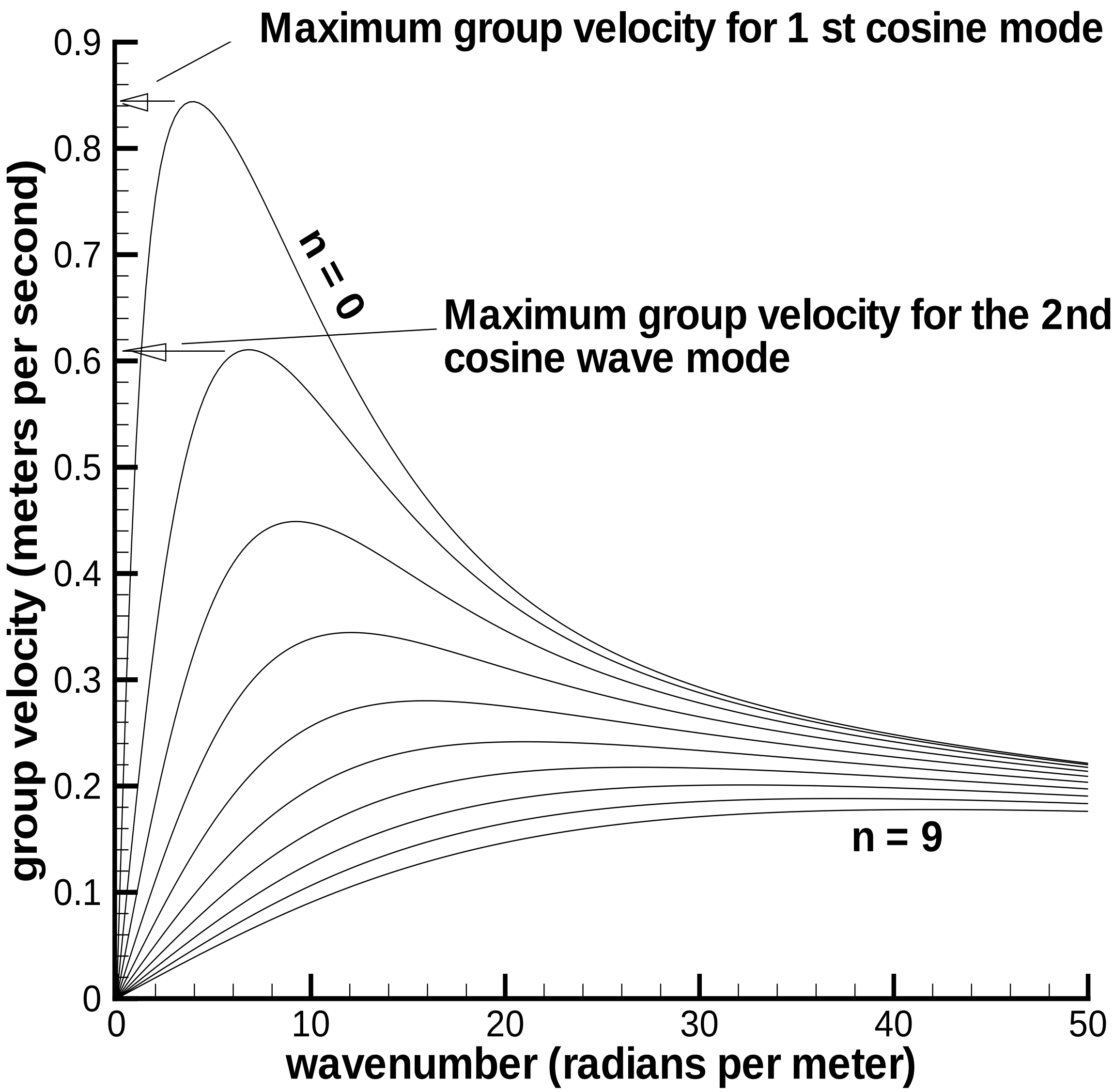

Figure 5: Group velocity as a function of the wavenumber k for various values of the eigenvalue k_{zn} and water depth $h = 0.1$ meters. The maximum group velocity for the first cosine wave mode ($C_{g,max}$) is indicated in the figure. Maximum group velocity for the second cosine mode also indicated in the figure.

cosine mode in the z direction. The excitation frequency is exactly at the *cut-off frequency*. Again, the wave maker starts from rest to the harmonic motion, and initially all wave frequencies are excited. A transient develops and propagates along the channel. The transient has a wave front which propagates with the maximum group velocity possible $C_{g,max}$ for the first cosine wave mode. Behind the transient we are left with the first cosine wave mode, since it is the only wave mode excited by the wave maker. The group velocity of this wave mode at its *cut-off frequency* is zero, so there is no energy propagation along the channel after the transient part of the solution is already far from the wave maker. Since the energy cannot be radiated away from the wave maker, we see the wave amplitude growing with time close to the wave maker. The cosine wave mode resonates with the wave maker in this case. To see the animation associated with this example, click here.

- Now the wave maker is a liner function in the z direction ($F(z) = z$). We show the evolution of the disturbance due to the action of the wave maker. We consider all modes that take part in the solution. We actually consider only a finite number of sine and cosine wave modes. As the wavenumber k_{zm} or k_{zn} associated with a wave mode increases, its amplitude decreases, so only a finite number of wave modes are significant. Again, the wave maker starts from rest to the harmonic motion. We have initially a transient which propagates along the channel. It has a wave front which propagates with the maximum possible group velocity, which is the maximum group velocity for the first sine wave mode. Ahead of the wave front ($x > C_{g,max}t$ for a given instant t, where $C_{g,max}$ is the maximum group velocity for the first sine wave mode) we have no waves disturbance. For a given instant t, the transient stays in the region $C_{g,max}t > x > C_g(\omega)t$, where $C_g(\omega)$ is the group velocity of the first sine wave mode at the excitation frequency ω. Behind this region we have the steady state solution. To see the animation associated with this example, click here.