General Method for Deriving an Element Stiffness Matrix

step I: select suitable displacement function

beam likely to be polynomial with one unknown coefficient for each (of four) degrees of freedom

$$\text{dof} = \delta = \begin{pmatrix} v_1 \\ v'_1 \\ v_2 \\ v'_2 \end{pmatrix} \quad \text{in matrix notation:}$$

say ...

$$v(x) = C_1 + x\cdot C_2 + x^2\cdot C_3 + x^3\cdot C_4$$

$$C := \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} \qquad H(x) := \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} \qquad v(x) := H(x)\cdot C \qquad 5.3.6$$

$$v(x) \rightarrow C_1 + x\cdot C_2 + x^2\cdot C_3 + x^3\cdot C_4 \qquad \frac{d}{dx}v(x) \rightarrow C_2 + 2\cdot x\cdot C_3 + 3\cdot x^2\cdot C_4$$

$$\delta(x) := \begin{pmatrix} v(x) \\ \frac{d}{dx}v(x) \end{pmatrix} \quad \delta(x) \rightarrow \begin{pmatrix} C_1 + x\cdot C_2 + x^2\cdot C_3 + x^3\cdot C_4 \\ C_2 + 2\cdot x\cdot C_3 + 3\cdot x^2\cdot C_4 \end{pmatrix} \quad \delta(0) \rightarrow \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \quad \delta(L) \rightarrow \begin{pmatrix} C_1 + L\cdot C_2 + L^2\cdot C_3 + L^3\cdot C_4 \\ C_2 + 2\cdot L\cdot C_3 + 3\cdot L^2\cdot C_4 \end{pmatrix}$$

in matrix form: for manipulation

$$\delta(x) = \begin{pmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \end{pmatrix}\cdot\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} \qquad \delta\_over\_C(x) := \begin{pmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \end{pmatrix} \qquad 5.3.7$$

step II: relate general displacements within element to its nodal displacement

$$\delta\_over\_C(0) \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \delta\_over\_C(L) \rightarrow \begin{pmatrix} 1 & L & L^2 & L^3 \\ 0 & 1 & 2\cdot L & 3\cdot L^2 \end{pmatrix}$$

in single matrix form:

$$\delta\_nodes = \begin{pmatrix} v_1 \\ v_{1\_p} \\ v_2 \\ v_{2\_p} \end{pmatrix} \qquad \text{define A such that} \qquad \delta\_nodes = A\cdot C$$

form by stacking node 1 with node 2

$$A := \text{stack}(\delta\_over\_C(0), \delta\_over\_C(L)) \quad A \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2\cdot L & 3\cdot L^2 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & L & 3\cdot L^2 \end{pmatrix}$$

$$\delta\_nodes = A\cdot C \quad \Rightarrow \quad C := A^{-1}\cdot\delta\_nodes \qquad A^{-1} \rightarrow \frac{1}{L^4}\cdot\begin{pmatrix} L^4 & 0 & 0 & 0 \\ 0 & L^4 & 0 & 0 \\ -3\cdot L^2 & -2\cdot L^3 & 3\cdot L^2 & -L^3 \\ 2\cdot L & L^2 & -2\cdot L & L^2 \end{pmatrix} \qquad 5.3.8b$$

$v(x) := H(x)\cdot C \qquad H(x) \rightarrow \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} \qquad v(x) := H(x)\cdot A^{-1}\cdot \delta\_nodes$ 5.3.9a

$$\frac{v(x)}{\delta\_nodes} \text{ simplify } \rightarrow \begin{pmatrix} \frac{L^3 - 3\cdot x^2\cdot L + 2\cdot x^3}{L^3} & x\cdot\frac{L^2 - 2\cdot x\cdot L + x^2}{L^2} & -x^2\cdot\frac{-3\cdot L + 2\cdot x}{L^3} & x^2\cdot\frac{-L + x}{L^2} \end{pmatrix}$$

shape function defined $N(x) := H(x)\cdot A^{-1}$ with $\xi = \frac{x}{L}$ => $x := \xi\cdot L$

$N(x) \rightarrow \begin{pmatrix} 1 - 3\cdot\xi^2 + 2\cdot\xi^3 & \xi\cdot L - 2\cdot\xi^2\cdot L + \xi^3\cdot L & 3\cdot\xi^2 - 2\cdot\xi^3 & -\xi^2\cdot L + \xi^3\cdot L \end{pmatrix}$ 5.3.9b although text has mix of ξ and x

Step III: express the internal deformation in terms of the nodal displacement

area resets x, redefines C, H and v

our problem is one of solid mechanics ; plane elasticity
deformation is strain: du/dx,
bending curvature d^2u/dx^2. v_2pr = d^2u/dx^2.

$v(x) \rightarrow C_1 + x\cdot C_2 + x^2\cdot C_3 + x^3\cdot C_4 \qquad \frac{d^2}{dx^2}v(x) \rightarrow 2\cdot C_3 + 6\cdot x\cdot C_4 \qquad v\_2pr(x) := (0\ \ 0\ \ 2\ \ 6\cdot x)\cdot C$

$v\_2pr(x) := (0\ \ 0\ \ 2\ \ 6\cdot x)\cdot A^{-1}\cdot\delta\_nodes \qquad B(x) := (0\ \ 0\ \ 2\ \ 6\cdot x)\cdot A^{-1} \qquad v\_2pr(x) := B(x)\cdot\delta\_nodes$

$$B(x) \rightarrow \begin{pmatrix} \frac{-6}{L^2} + 12\cdot\frac{x}{L^3} & \frac{-4}{L} + 6\cdot\frac{x}{L^2} & \frac{6}{L^2} - 12\cdot\frac{x}{L^3} & \frac{-2}{L} + 6\cdot\frac{x}{L^2} \end{pmatrix}$$ 5.3.10

step IV: express the internal force in terms of the nodal displacement

the "internal force" is the bending moment and
as with internal deformation,
this is a problem in bending so the relationship is

$$M(x) = E\cdot I\cdot\frac{d^2}{dx^2}v(x) = E\cdot I\cdot v\_2pr(x)$$

we just developed $v\_2pr(x) := B(x)\cdot\delta\_nodes$

copy here for use elsewhere:

$$B(x) := \begin{pmatrix} \frac{-6}{L^2} + 12\cdot\frac{x}{L^3} & \frac{-4}{L} + 6\cdot\frac{x}{L^2} & \frac{6}{L^2} - 12\cdot\frac{x}{L^3} & \frac{-2}{L} + 6\cdot\frac{x}{L^2} \end{pmatrix}$$

$$v\_2pr(x) \rightarrow \left[\left(\frac{-6}{L^2} + 12\cdot\frac{x}{L^3}\right)\cdot\delta\_nodes \quad \left(\frac{-4}{L} + 6\cdot\frac{x}{L^2}\right)\cdot\delta\_nodes \quad \left(\frac{6}{L^2} - 12\cdot\frac{x}{L^3}\right)\cdot\delta\_nodes \quad \left(\frac{-2}{L} + 6\cdot\frac{x}{L^2}\right)\cdot\delta\_nodes\right]$$

define:

$$M(x) := E\cdot I\cdot v\_2pr(x) \qquad bb := \frac{E\cdot I}{L^3} \qquad \frac{M(x)}{bb\cdot\delta\_nodes} \text{ simplify } \rightarrow [-6\cdot L + 12\cdot x \ \ 2\cdot(-2\cdot L + 3\cdot x)\cdot L \ \ 6\cdot L - 12\cdot x \ \ 2\cdot(-L + 3\cdot x)\cdot L]$$

5.3.12a

since M(x) is linear, we can calculate M(x) at the nodes (N.B. M is the *internal* moment)
define S

$$\begin{pmatrix} M(0) \\ M(L) \end{pmatrix} = S \cdot \delta\_nodes \qquad \text{5.3.12b}$$

$$S\_over\_bb = \begin{pmatrix} \dfrac{M(0)}{bb \cdot \delta\_nodes} \\ \dfrac{M(L)}{bb \cdot \delta\_nodes} \end{pmatrix}$$

$$\frac{M(0)}{bb \cdot \delta\_nodes} \rightarrow \begin{pmatrix} -6 \cdot L & -4 \cdot L^2 & 6 \cdot L & -2 \cdot L^2 \end{pmatrix}$$

$$\frac{M(L)}{bb \cdot \delta\_nodes} \rightarrow \begin{pmatrix} 6 \cdot L & 2 \cdot L^2 & -6 \cdot L & 4 \cdot L^2 \end{pmatrix}$$

substituting bb = E*I/L^3

note that this is similar to M1 and M2 with sign reversal in top element

$$S = \frac{E \cdot I}{L^3} \begin{bmatrix} \begin{pmatrix} -6 \cdot L & -4 \cdot L^2 & 6 \cdot L & -2 \cdot L^2 \end{pmatrix} \\ \begin{pmatrix} 6 \cdot L & 2 \cdot L^2 & -6 \cdot L & 4 \cdot L^2 \end{pmatrix} \end{bmatrix}$$

step V: obtain the element stiffness matrix ke by relating nodal forces to nodal displacements

we will do this by the principle of virtual work:
assume an arbitrary virtual nodal displacement:

$$\delta\_star := \begin{pmatrix} v1\_star \\ v1'\_star \\ v2\_star \\ v2'\_star \end{pmatrix}$$

actual nodal forces are:

$$f := \begin{pmatrix} f1 \\ M1 \\ f2 \\ M2 \end{pmatrix}$$

external virtual work is force * virtual deflection: $W_{ext} := \delta\_star^T \cdot f$

$$W_{ext} \rightarrow v1\_star \cdot f1 + v1'\_star \cdot M1 + v2\_star \cdot f2 + v2'\_star \cdot M2$$

internal work = work done in imposing curvature on the beam:
for an arbitrary virtual curvature v''_star(x)

$$W_{int} := \int_0^L v\_2pr\_star(x)^T \cdot M(x)\, dx$$

$$M(x) = internal\_moment$$

using transpose as v''_star(x) is a scalar but will involve 4 x 1 vectors to multiply the scalar M(x) with vector components later.
if arbitrary virtual curvature v''_star(x) is imposed indirectly by virtual nodal displacement
v''_star(x) is related to the δ_star by B(x)

$$v\_2pr(x) := B(x) \cdot \delta\_nodes$$

from above

$$v\_2pr\_star(x) := B(x) \cdot \delta\_star$$

δ_star is understood to be nodal

and ... $$v\_2pr\_star(x)^T = (B(x) \cdot \delta\_star)^T = \delta\_star^T \cdot B(x)^T$$

now using $$M(x) = E \cdot I \cdot \frac{d^2}{dx^2} v(x) = E \cdot I \cdot v\_2pr(x) \qquad v\_2pr(x) := B(x) \cdot \delta\_nodes \qquad M(x) = E \cdot I \cdot B(x) \cdot \delta\_nodes$$

$$W_{int} = \int_0^L v\_2pr\_star(x)^T \cdot M(x)\, dx = \int_0^L \delta\_star^T \cdot B(x)^T \cdot E \cdot I \cdot B(x) \cdot \delta\_nodes\, dx$$

taking the constants outside the integral and equating internal to external work the constants have to come out of the correct side to maintain matrix math

$$W_{ext} = \delta\_star^T \cdot f = W_{int} = \delta\_star^T \cdot E \cdot I \int_0^L B(x)^T \cdot B(x)\,dx \cdot \delta\_nodes$$

cancelling δ_star =>

$$f = E \cdot I \int_0^L B(x)^T \cdot B(x)\,dx \cdot \delta\_nodes = k_e \cdot \delta\_nodes$$

what we came for

$$k_e = E \cdot I \cdot \int_0^L B(x)^T \cdot B(x)\,dx$$

$$B(x)^T \rightarrow \begin{pmatrix} \frac{-6}{L^2} + 12 \cdot \frac{x}{L^3} \\ \frac{-4}{L} + 6 \cdot \frac{x}{L^2} \\ \frac{6}{L^2} - 12 \cdot \frac{x}{L^3} \\ \frac{-2}{L} + 6 \cdot \frac{x}{L^2} \end{pmatrix}$$

$$B(x) \rightarrow \begin{pmatrix} \frac{-6}{L^2} + 12 \cdot \frac{x}{L^3} & \frac{-4}{L} + 6 \cdot \frac{x}{L^2} & \frac{6}{L^2} - 12 \cdot \frac{x}{L^3} & \frac{-2}{L} + 6 \cdot \frac{x}{L^2} \end{pmatrix}$$

all we need is $\int_0^L B(x)^T \cdot B(x)\,dx$

(it won't compute symbolicly so I wrote it out in the collapsed area)

---

result; copied from rhs

$$\int_0^L B(x)^T \cdot B(x)\,dx = \begin{pmatrix} \frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L} \\ \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L} \end{pmatrix}$$

$$k_e = E \cdot I \cdot \int_0^L B(x)^T \cdot B(x)\,dx = E \cdot I \cdot \begin{pmatrix} \frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L} \\ \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L} \end{pmatrix}$$

so ...

$$\begin{pmatrix} f_{y1} \\ M1 \\ f_{y2} \\ M2 \end{pmatrix} = \frac{E \cdot I}{L^3} \cdot \begin{pmatrix} 12 & 6 \cdot L & -12 & 6 \cdot L \\ 6 \cdot L & 4 \cdot L^2 & -6 \cdot L & 2 \cdot L^2 \\ -12 & -6 \cdot L & 12 & -6 \cdot L \\ 6 \cdot L & 2 \cdot L^2 & -6 \cdot L & 4 \cdot L^2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}$$

we now have $f = k_e \cdot \delta$