Part 1

Massachusetts Institute of Technology
Department of Mechanical Engineering

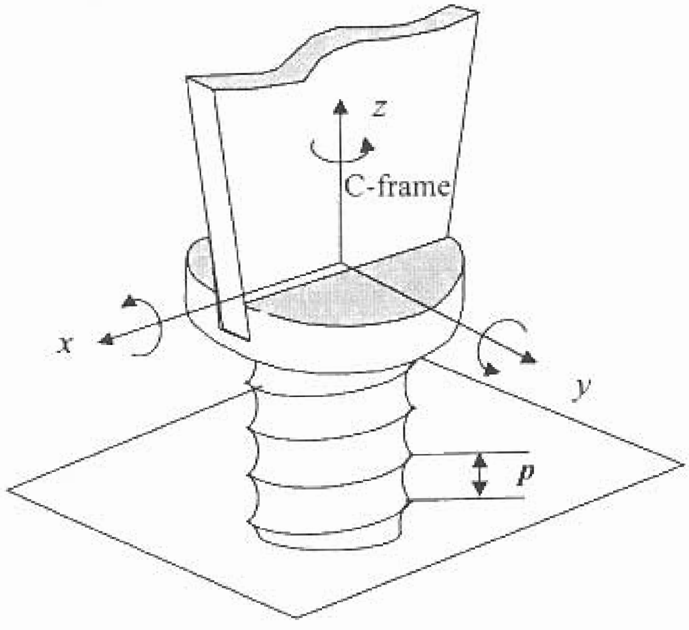
2.12 Introduction to Robotics
Exercise Problems for the End-of-Term Examination

Problem 1

A carpenter robot is fixing a screw with a screwdriver, as shown in the figure below. Assume that the task process is quasi-static and friction-less. Use the C-frame shown in the figure. Note that the robot, although not shown, holds the screwdriver. At the instant shown, the y-axis of the C-frame is perpendicular to the groove of the screw head. The pitch of the screw is p, and the screw head is tightly mated with the screwdriver, having no play in the y direction. Answer the following questions.

a). Translational and rotational velocities in the z direction are coupled to each other due to the screw. Consider the $v_z - \omega_z$ plane shown below in order to describe the directions of constraint space and admissible motion space. Note that v_a is in the direction of admissible motion space while v_c is in the direction of constraint space. Obtain the components of the vectors pointing in the directions of v_a and v_c.

b). Obtain the natural and artificial constraints for the robot to insert the screw based on hybrid position and force control. Use v_a and v_c as well as associated force/torque f_a and f_c for describing the natural and artificial constraints.



The sketch below was intentionally made inaccurate.

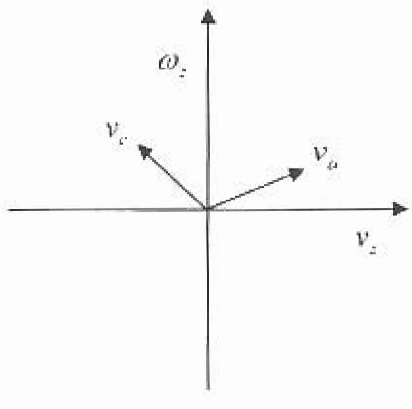


Figure 1 Screw with C-frame and $v_z - \omega_z$ plane

Problem 2

A two degree-of-freedom robot with one prismatic joint is shown below. Note that the prismatic joint is in the direction of angle ϕ from the centerline of the first link. As shown in the figure, joint angle θ and distance z between the tip of the first link and the mass centroid of the second link are used as generalized coordinates. The first actuator fixed to the base produces torque τ about the first joint, while the second actuator located at the tip of the first link generates linear force f acting on the second link. Using the parameters shown in the figure, answer the following questions.

a) Obtain the moment of inertia seen by the first joint when the second joint is fixed at z.

b) Sketch free-body diagrams of the individual links and obtain Newton-Euler equations. Ignore friction.

c) Eliminate the constraint force and moment between the two links, and obtain the closed-form equations of motion with respect to generalized coordinates, θ and z.

d) When the first joint is rotating at a constant speed $\dot{\theta} = \omega_0$, the second link moves at a constant linear velocity $\dot{z} = V_0$. Obtain the Coriolis force induced in the second link and show in which direction the Coriolis force is acting. Obtain the torque acting on the first joint due to the Coriolis force, and show that this torques agrees with the corresponding nonlinear term in the equations of motion.

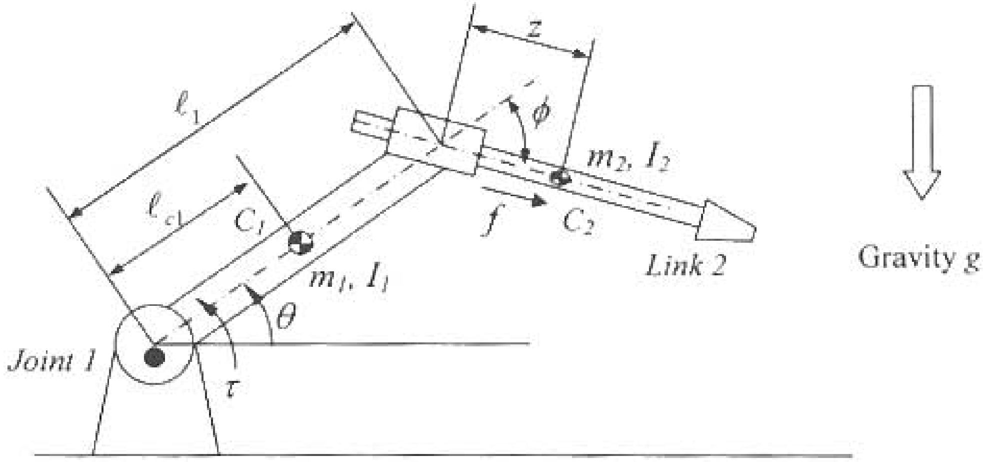


Figure 1

e) We want to decouple and linearize the system with respect to the generalized coordinates, θ and z, and form independent single-input-single-output PD control loops as shown in Figure 2. Obtain the computed torque control law to achieve this control and show the block diagram of the control system, including both the computed control loops and the PD control loops.

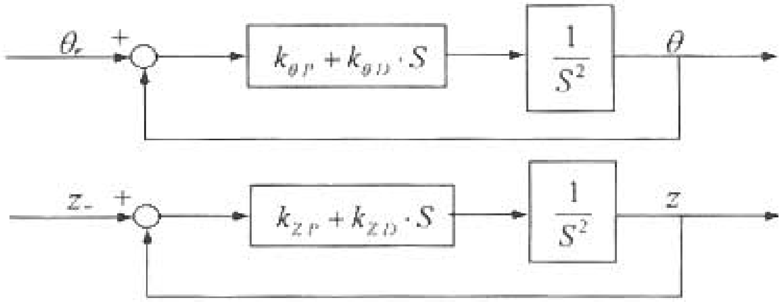


Figure 2

Massachusetts Institute of Technology
Department of Mechanical Engineering

2.12 Introduction to Robotics
End-of-Term Examination
Solutions
December 1, 2004

Problem 1

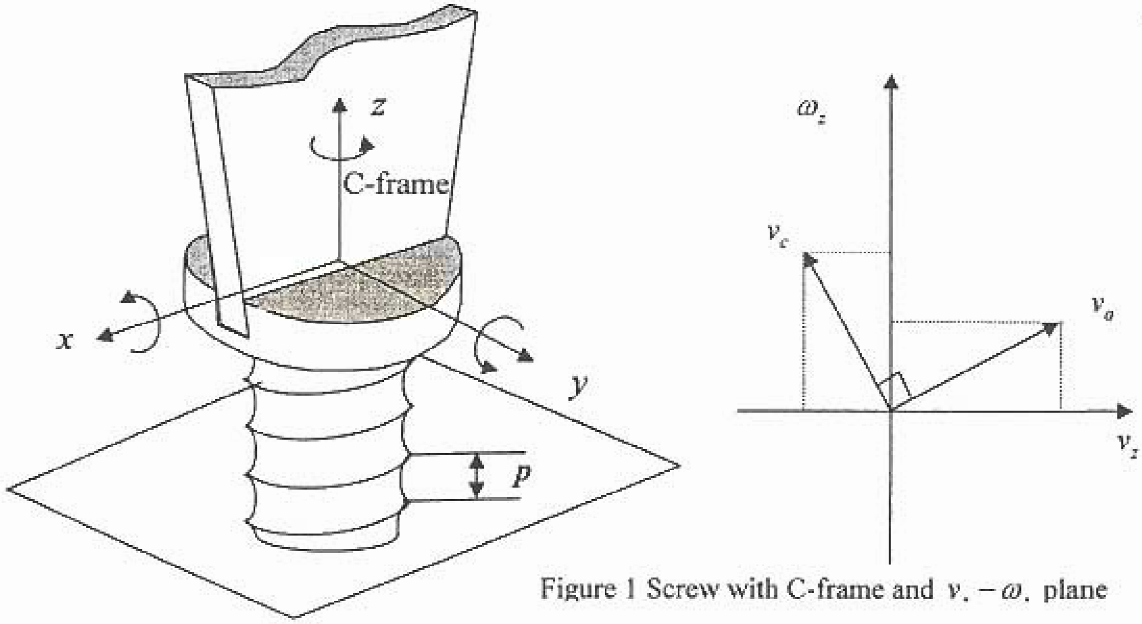


Figure 1 Screw with C-frame and $v_z - \omega_z$ plane

a). v_z and ω_z are related : $v_z = \frac{p}{2\pi}\omega_z$, or $v_z : \omega_z = p : 2\pi$.
This means that in the $v_z - \omega_z$ plane the direction of admissible motion, denoted by vector $\mathbb{a}$ is

$$\mathbb{a} = \begin{bmatrix} p \\ 2\pi \end{bmatrix}$$

The constraint space is orthogonal to $\mathbb{a}$. Therefore the direction denoted by vector $\mathbb{C}$ is

$$\mathbb{C} = \begin{bmatrix} -2\pi \\ p \end{bmatrix}$$

Note that

$$\mathbb{a}^T \mathbb{C} = \begin{bmatrix} p \\ 2\pi \end{bmatrix}^T \begin{bmatrix} -2\pi \\ p \end{bmatrix} = 0 \quad \therefore \quad \mathbb{a} \perp \mathbb{C}.$$

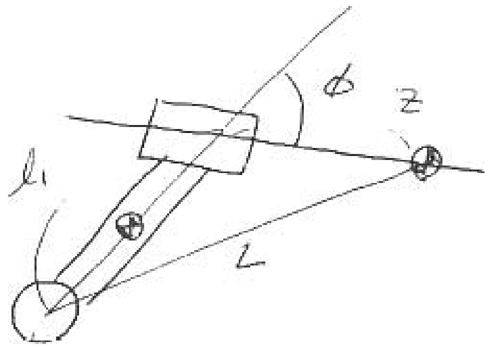
1-b). Now, instead of v_z and ω_z, let us use v_a and v_c as new coordinates for describing the natural and artificial constraints. Good thing, axes v_a and v_c are completely aligned with the directions of admissible motion and constraint. Unlike v_z and ω_z, admissible motion space and constraint space are <u>not mixed</u>.

	Kinematic		Static	
Natural Constraint	$v_y = 0$, $\omega_x = 0$, $\omega_y = 0$	$v_c = 0$	$f_z = 0$	$f_a = 0$
Artificial Constraint	$v_x = 0$	$v_a = -V < 0$	$f_y = 0$, $\tau_x = 0$, $\tau_y = 0$	$f_c = -F < 0$

Note f_a and f_c are forces (in fact, generalized forces) corresponding to v_a and v_c, respectively.

Problem 2

a)

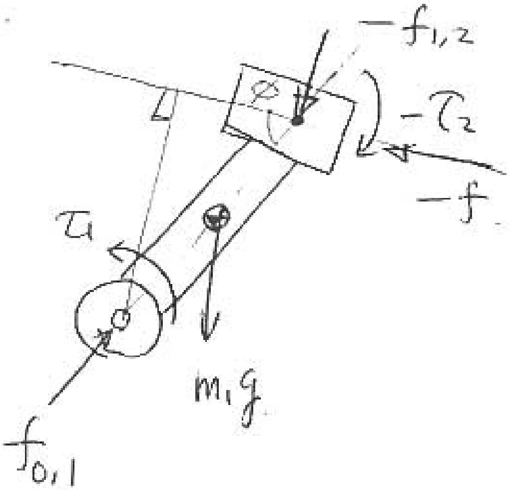


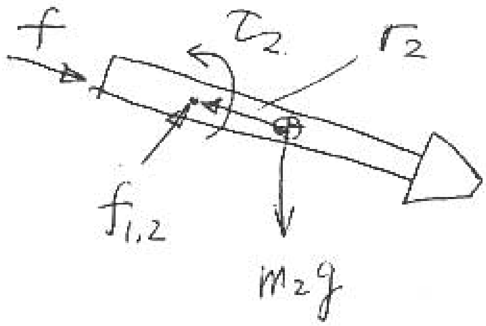
$$H_{11} = I_1 + m_1 l_{c1}^2 + I_2 + m_2 L^2$$

$$L^2 = l_1^2 + z^2 - 2 l_1 z \cos(\pi - \phi)$$
$$= l_1^2 + z^2 + 2 l_1 z \cos\phi$$

$$\therefore H_{11} = I_1 + m_1 l_{c1}^2 + I_2 + m_2 \left(l_1^2 + z^2 + 2 l_1 z \cos\phi \right)$$

b)





Rotation about a fixed point

$$(I_1 + m_1 l_{c1}^2)\ddot{\theta} = \tau_1 - \tau_2 + l_1 \begin{pmatrix} c \\ s \end{pmatrix} \times (-f_{1,2}) + l_1 \sin\phi \, f$$
$$- m_1 g \, l_{c1} \cos\theta \quad \cdots \text{Link 1}$$

Link 2

$$I_2 \ddot{\theta} = \tau_2 + r_2 \times f_{1,2}$$

$$m_2 \ddot{x}_{c2} = f_{1,2} + \begin{pmatrix} 0 \\ -m_2 g \end{pmatrix} + f \begin{pmatrix} \cos(\theta - \phi) \\ \sin(\theta - \phi) \end{pmatrix}$$

$$r_2 = -z \begin{pmatrix} \cos(\theta - \phi) \\ \sin(\theta - \phi) \end{pmatrix}$$

c)

Find $\ddot{x}_{c2} = \left(l_1\ddot{\theta}\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix} - l_1\dot{\theta}^2\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + \ddot{z}\begin{pmatrix}\cos\theta-\varphi\\ \sin\theta-\varphi\end{pmatrix} + \dot{z}\dot{\theta}\begin{pmatrix}-\sin\theta-\varphi\\ \cos\theta-\varphi\end{pmatrix} - z\dot{\theta}^2\begin{pmatrix}\cos\theta-\varphi\\ \sin\theta-\varphi\end{pmatrix} + (\dot{z}\dot{\theta}+z\ddot{\theta})\begin{pmatrix}-\sin\theta-\varphi\\ \cos\theta-\varphi\end{pmatrix}\right)$

Note that $f_{12} = f_c\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + f\begin{pmatrix}\cos\theta-\varphi\\ \sin\theta-\varphi\end{pmatrix}$ premultiply by $(\sin\theta, -\cos\theta)$

$$(\sin\theta, -\cos\theta)f_{12} = f(\sin\theta, -\cos\theta)\begin{pmatrix}\cos\theta-\varphi\\ \sin\theta-\varphi\end{pmatrix} = f\left(\sin\theta\cos\theta-\varphi - \cos\theta\sin\theta-\varphi\right)$$

$$\therefore (\sin\theta, -\cos\theta)f_{12} = f\sin\varphi$$

Recall for link 2

$$f_{12} = m_2\ddot{x}_{c_2} + \begin{pmatrix}0\\ m_2 g\end{pmatrix} + -f\begin{pmatrix}\cos\theta-\varphi\\ \sin\theta-\varphi\end{pmatrix}$$

$$\cdot\ (\sin\theta, -\cos\theta)\left\{m_2\ddot{x}_{c_2} + \begin{pmatrix}0\\ m_2 g\end{pmatrix} - f\begin{pmatrix}\cos\theta-\varphi\\ \sin\theta-\varphi\end{pmatrix}\right\} = f\sin\varphi$$

$$\cdot\ m_2\left(-l_1\ddot{\theta} + \ddot{z}\sin\varphi + \dot{z}\dot{\theta}(-\cos\varphi) + z\dot{\theta}^2(\sin\varphi) + (\dot{z}\dot{\theta} + z\ddot{\theta})(-\cos(\varphi))\right)$$

$$+ m_2 g\cos\theta = 2f\sin\varphi$$

$$\Rightarrow f = m_2\left\{\frac{-l_1\ddot{\theta}}{2\sin\varphi} - \frac{z\cos\varphi\,\ddot{\theta}}{2\sin\varphi} + \frac{\ddot{z}}{2} - \frac{\cos\varphi}{\sin\varphi}\dot{z}\dot{\theta} + \frac{z\dot{\theta}^2}{2} + \frac{g\cos\theta}{2\sin\varphi}\right\}$$

Now, find τ_1;

$$(I_1 + I_2 + m_1 l_{c1}^2)\ddot{\theta} = \tau_1 + r_2\times f_{12} + l_1\begin{pmatrix}s\\ c\end{pmatrix}\times(-f_{12}) + l_1\sin\varphi f - m_1 g l_{c1}\cos\theta$$

compute $r_2\times f_{12}$

$$= -z\begin{pmatrix}\cos\theta-\varphi\\ \sin\theta-\varphi\end{pmatrix}\times\left(m_2\ddot{x}_{c_2} + \begin{pmatrix}0\\ m_2 g\end{pmatrix}\right) - f\begin{pmatrix}\cos\theta-\varphi\\ \sin\theta-\varphi\end{pmatrix}$$

$$= -zm_2\left(l_1\ddot{\theta}\cos\varphi - l_1\dot{\theta}^2\sin\varphi + 2\dot{z}\dot{\theta} + z\ddot{\theta}\right) - m_2 g z\cos\theta-\varphi$$

compute $l_1\begin{pmatrix}s\\ c\end{pmatrix}\times(-f_{12})$

$$= -l_1\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\times\left(m_2\ddot{x}_{c_2} + \begin{pmatrix}0\\ m_2 g\end{pmatrix} - f\begin{pmatrix}\cos\theta-\varphi\\ \sin\theta-\varphi\end{pmatrix}\right)$$

$$= -\Big\{l_1 m_2\big(-l_1\dot{\theta}^2\cos 2\theta - \ddot{z}\cos 2\theta-\varphi + \dot{z}\dot{\theta}(\sin 2\theta-\varphi)2 + z\dot{\theta}^2\cos 2\theta-\varphi$$

$$+ z\ddot{\theta}\sin 2\theta-\varphi + l_1 m_2 g\cos\theta-\varphi - \frac{m_2 l_1^2\cos 2\theta-\varphi\,\ddot{\theta}}{2\sin\varphi} + \frac{m_2 l_1}{2}\cos 2\theta-\varphi\,\ddot{z}$$

$$- \frac{m_2 l_1\cos\varphi\cos 2\theta-\varphi}{\sin\varphi}\dot{z}\dot{\theta} - \frac{m_2\sin\varphi\, l_1\cos 2\theta-\varphi}{2\sin\varphi}z\dot{\theta}^2 - \frac{m_2 l_1\cos\varphi\cos 2\theta-\varphi}{2\sin\varphi}z\ddot{\theta}$$

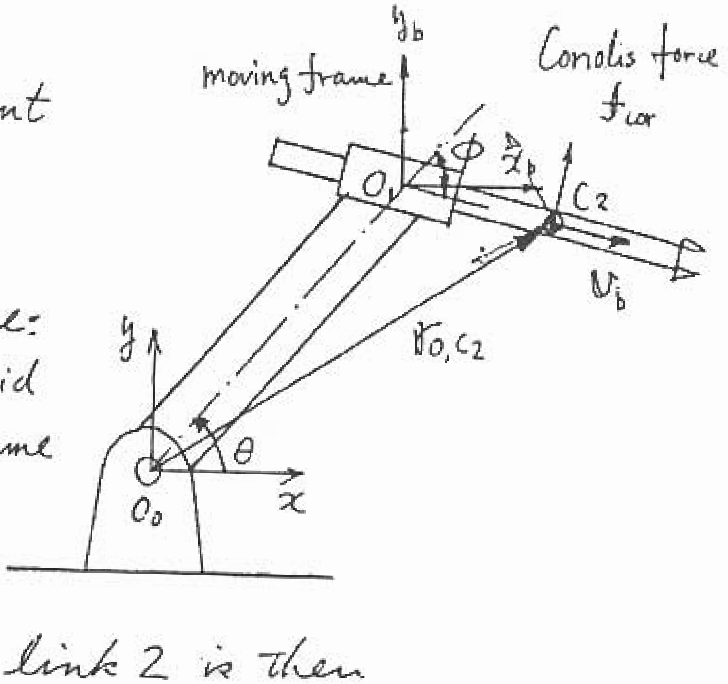
$$+ m_2 g l_1\cos\theta\cos 2\theta-\varphi\Big\}$$

$$\therefore\ \tau_1 = (I_1 + I_2 + m l_{c_1}^2)\ddot{\theta} + (z m_2)(l_1 \cos\varphi + z)\ddot{\theta} - z m_2 l_1 \sin\varphi\, \dot{\theta}^2 + 2 z m_2 \dot{z}\dot{\theta} + m_2 g z \cos\theta\text{-}\varphi$$
$$+ (-l_1^2 m_2 \cos 2\theta\, \dot{\theta}^2) - l_1 m_2 \cos 2\theta\text{-}\varphi\, \ddot{z} + 2 l_1 m_2 \sin 2\theta\text{-}\varphi\, \dot{z}\dot{\theta} + l_1 m_2 \cos 2\theta\text{-}\varphi\, z\dot{\theta}^2$$
$$+ l_1 m_2 \sin 2\theta\text{-}\varphi\, z\ddot{\theta} + l_1 m_2 g \cos\theta\text{-}\varphi - \frac{m_2 l_1^2 \cos 2\theta\text{-}\varphi}{2\sin\varphi}\ddot{\theta} + \frac{m_2 l_1 \cos 2\theta\text{-}\varphi}{2}\ddot{z}$$
$$+ \left(-\frac{m_2 l_1 \cos 2\theta\text{-}\varphi}{\tan\varphi}\dot{z}\dot{\theta}\right) - \frac{m_2 l_1 \cos 2\theta\text{-}\varphi}{2} z\dot{\theta}^2 - \frac{m_2 l_1 \cos 2\theta\text{-}\varphi}{\tan\theta} z\ddot{\theta}$$
$$+ m_2 g l_1 \cos\theta \cos 2\theta\text{-}\varphi$$

1-3)

d) Consider a moving frame fixed to link 1. At the instant shown in the figure, the moving frame $O_1 - x_b y_b$ is aligned with the base frame $O_0 - xy$. The velocity of centroid C_2 relative to the moving frame is

$$v_b = \begin{bmatrix} V_0 \cos(\theta - \phi) \\ V_0 \sin(\theta - \phi) \end{bmatrix}$$



The Coriolis force induced in link 2 is then

$$f^*_{cor} = 2 m_2 \omega \times v_b^* = 2 m_2 \omega_0 V_0 \begin{pmatrix} -\sin(\theta - \phi) \\ \cos(\theta - \phi) \\ 0 \end{pmatrix}.$$

where $\omega = (0, 0, \omega_0)^T$ and $v_0^* = \begin{pmatrix} v_0 \\ 0 \end{pmatrix}$ are 3×1 vectors.

The x.y components of f^*_{cor} shows that the Coriolis force acts in the direction perpendicular to the center line of link 2 as shown in the figure above.

The torque about the first joint axis generated by the Coriolis force is

$$\tau_{cor} = r_{o,c2} \times f_{cor} = \begin{vmatrix} l_1\cos\theta + z\cos(\theta-\phi) , & -\sin(\theta-\phi) \\ l_1\sin\theta + z\sin(\theta-\phi), & \cos(\theta-\phi) \end{vmatrix} \cdot 2m_2\omega_0 V_0$$

$$\tau_{cor} = 2(l_1\cos\phi + z)\, m_2 \omega_0 V_0$$

This agrees with the third term in the first equation of motion ①, when $\dot{z} = V_0$ and $\dot{\theta} = \omega_0$.

e)

$$\begin{cases} u_\theta =: k_{\theta p}(\theta_r - \theta) + k_{\theta d}(\dot{\theta}_r - \dot{\theta}) \\ u_z =: k_{zp}(z_r - z) + k_{zd}(\dot{z}_r - \dot{z}) \end{cases}$$ Individual PD Controls

$$\begin{cases} \tau =: H_{11}(z)\cdot u_\theta + H_{12} u_z + h_1(z)\dot{z}\dot{\theta} + G_1(\theta, z) \\ f =: H_{12} u_\theta + H_{22} u_z + h_2(z)\dot{\theta}^2 + G_2(\theta) \end{cases}$$ Computed Control Laws.