MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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2.29 NUMERICAL FLUID MECHANICS — SPRING 2015

Problem Set 5

Issued: Wednesday, April 8, 2015 Due: Monday, April 27, 2015

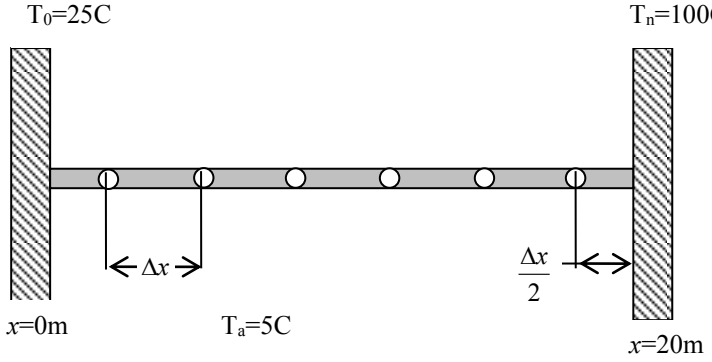
Grading Note: Please provide your solutions either as hand-written/hard-copy solutions or by submitting via course website . MATLAB® codes should be submitted via course website . The bulk of the grades will be given to detailed explanations and to algorithms and numerical schemes that capture the essence of the numerical problems. We know that successful coding of numerical schemes can be time consuming and prone to small errors. Such small errors or omissions in a code will not be heavily penalized.

The goals of this Problem Set are to: (i) learn about the set-up of finite-volume methods and their applications, with comparisons to finite-difference schemes; (ii) compare and utilize discrete advection fluxes including more advanced solvers for 1D nonlinear convective problems; (iii) apply, evaluate and compare properties of Navier-Stokes solvers for 2D incompressible flows.

Problem 1: ***Problem 5 of Problem Set II (1D Fin) with a control volume approach.***
Consider the following PDE derived from a heat balance for a long thin fin (see figure).

$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0$$

where T is temperature (°C), x the distance along the rod (m), h' a heat transfer coefficient between the fin and the ambient air (m^{-2}) and T_a is the temperature of the surrounding air (°C). This is the steady 'fin equation' for convective cooling of an extended surface.



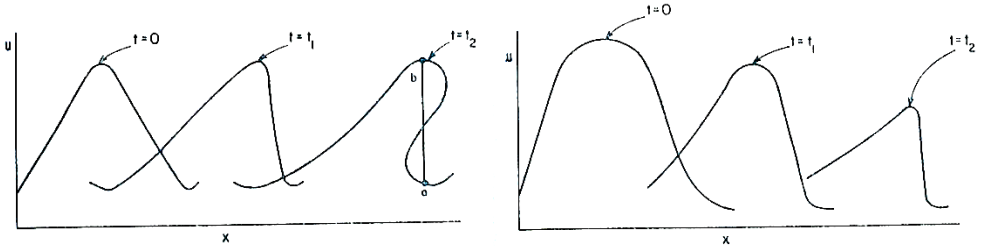
a) Divide the fin into a set of control volumes surrounding the finite-difference nodes. Develop a control volume FD scheme for this equation within the interior nodes, using "central differencing" in space (piece-wise quadratic shape function). Very briefly, interpret the physical meaning of each of the flux terms in the discretized equation.
b) Develop the discretized boundary condition at the two boundary nodes (located at Δx/2 from each of the two vertical plates).
c) Briefly compare the schemes a) and b) to those obtained in problem 5 of Problem Set II.
d) Set-up these equations a) and b) into a linear algebraic system **A x** = **b**. Taking advantage of the structure of the matrix **A** you obtain, solve the resulting linear algebraic problem in MATLAB for the parameter values given in part d) of problem 5 of Problem Set II. To do this,

you can directly utilize the MATLAB codes that have been given to you in problem sets or lecture, or in even MATLAB itself.
e) Optional (Bonus): only for the curious, assume that $h=h'(x)$ varies with x and derive the corresponding FV scheme you would obtain in c). Compare this scheme to the finite-difference scheme you would have obtained in problem 5 of Problem Set II if $h=h'(x)$ varied with x.

Problem 2: Burger's equation. Burger's equation is often referred to as the "poor man's Navier-Stokes equation". It is employed to illustrate idealized nonlinear fluid processes (e.g. steepening of waves, turbulence), and it is frequently utilized to evaluate new schemes for handling non-linear fluxes. It is given by:

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial (u^2)}{\partial x}=0\,.$$

a) Using the chain-rule, evaluate the spatial derivative, and compare the re-written equation to the Sommerfeld Wave equation $\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$. From your answer, discuss how you expect the solution to behave for: *i)* $u(x,t=0)>0$, *ii)* $u(x,t=0)<0$.
Some fluid solutions idealized by Burger's equation are sketched here:



b) Develop a control volume FV scheme for this equation assuming uniform discretization of the interior nodes. Let u be a constant on each control volume. For the flux terms u^2 at the edges of control volumes, use *i*) Central fluxes, *ii*) Upwind fluxes, *iii*) QUICK fluxes, *iv*) following fluxes:

$$f_{j-1/2}=\left(\frac{u_j+u_{j-1}}{2}\right)^2+\max(|u_j|,|u_{j-1}|)\left(\frac{u_{j-1}-u_j}{2}\right),\quad f_{j+1/2}=\left(\frac{u_j+u_{j+1}}{2}\right)^2+\max(|u_j|,|u_{j+1}|)\left(\frac{u_j-u_{j+1}}{2}\right)$$

[These latter fluxes are a "Roe solver" which is a particular Riemann solver, for details see Leveque, 2002].

c) Implement your scheme on a <u>periodic</u> domain x=[0,1] with integration over the time interval T=[0,1]. Choose an appropriate Δx and Δt so that your solution is stable. Use the following initial conditions:

```
i)   ui=exp(-10*(x-0.5).^2)
ii)  ui=(0.25*sin(2*pi*x')+0.5*sin(4*pi*x')+0.75)
iii) ee=0.005; %"Look-alike" Solitons
     ui=( 1+0.5*(1-tanh((x'-0.3)/4/ee))-0.5*(1-tanh((x'-0.1)/4/ee)))+...
        (-1+0.5*(1-tanh((x'-0.7)/4/ee))-0.5*(1-tanh((x'-0.9)/4/ee)))
```

Do not save every timestep, but either plot or save your solution every few timesteps. In realistic applications, it is often not necessary to save every timestep, nor is it possible due to physical memory constraints.

d) Discuss your solutions (Hint: not all flux discretizations will "work"). Were your answers in a) correct? Which flux-discretization scheme gave the best answer? How good do you think these answers are (comment on numerical diffusion)? How do you think you could improve these results?

Problem 3: Numerical Solution of Navier-Stokes Equations. A MATLAB code (called the 2.29 finite volume MATLAB framework) for solving the Navier-Stokes equations was provided. The Navier-Stokes equations with Boussinesq approximation for buoyancy are as follows:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\nabla^2\mathbf{u} = -\nabla P^* + g\rho^*\hat{\mathbf{k}} + \mathbf{F}(t,\mathbf{x}) \qquad [1]$$

$$\nabla\cdot\mathbf{u} = 0 \quad [2] \qquad \frac{\partial \rho^*}{\partial t} + \mathbf{u}\cdot\nabla\rho^* - \kappa\nabla^2\rho^* = 0 \quad [3]$$

$$\text{where,} \quad P^* = \frac{P}{\rho_0}, \qquad \rho^* = \frac{\rho'}{\rho_0}, \qquad \rho' << \rho_0$$

The solvers used are based on several projection methods (For a review, see Guermond, J.L., Minev, P., and Shen, Jie. (2006). *An overview of projection methods for incompressible flows*. Comput. Methods Appl. Mech. Eng., 195:6011-6045).

In this problem you are asked to run the code, compare various schemes to each other, and tie together some of what you have learned in 2.29. To do this, you will have to use the "TestCase" function in the 2.29 finite volume MATLAB framework.

First, you need to run "setup.m" to add the paths of the source codes. Then type "help TestCase" from the MATLAB command window for a description. The first argument of the input to "TestCase" selects a particular test problem, whereas the second argument specifies a set of parameters via the structure variable "app":

```
function [P,u,v,rho,NodeP,Nodeu,Nodev,Tsw,Ssw] = TestCase(problem, app)
```

The function returns the solution at the final time step. Therefore, if you just want to view the plots and do not want the final solution, you can call "TestCase" without specifying output variables. For example:

```
>>TestCase(2,app);
```

For some questions, you may have to set-up the following entries in the structure variable "app":

```
% INPUTS:
%       Problem:    Test case number (see below)
%       app:        Application data structure. TestCase assigns
%                   default values. Can contain entries such as:
%       app.Nx:           Number of control volumes in x direction
%       app.Ny:           Number of control volumes in y direction
%       app.T:            Final time (Optional)
%       app.dt:           Timestep size (Optional)
%       app.PlotIntrvl:   Number of timesteps till solution is plotted
%       app.nu:           Viscosity used in momentum equations (Optional)
%                         app.nu = 1/Re
%       app.kappa:        Diffusivity used in Density equation (Optional)
%       app.NonInc:       If true (false by default), solves the equations
%                         using the non-incremental projection method
```

```
%       app.NonRot:       If true (false by default), solves the equations
%                         using the incremental projection method in the
%                         non-rotational form
%       app.Advect:       'TVD'   -- Use the Total Variation Diminishing
%                                    advection scheme (DEFAULT)
%                         'QUICK' -- Quadratic Upwind Interpolation scheme
%                                    for Convective Kinetics
%                         'UW'    -- Use the UpWind advection scheme
%                         'CDS'   -- Use the Central Differencing Scheme
```

a) Advection schemes: Run "TestCase(2)" and compare to TestCase(3). Compare the UPWIND and QUICK advection schemes. Include plots of the final tracer concentration in your report.

b) Lock-Exchange problem: A barrier initially separates two fluids, one denser than the other. This barrier is removed at the start of the simulation. Run "TestCase(6,app)" with app.Advect='UW', then app.Advect='QUICK' and then app.Advect='TVD'. Compare the three solutions obtained. As part of your answer, provide a plot for each of the three difference cases.

c) To test the convergence of this solver, a Lid driven cavity flow test case is given. Run "Cavity_Convergence_Test.m" and compare the graph to the following figure from chapter 7.8 of Ferziger and Peric (2002), for the same problem. Calculate the rate of convergence for the centerline velocity (using the finest resolution available).

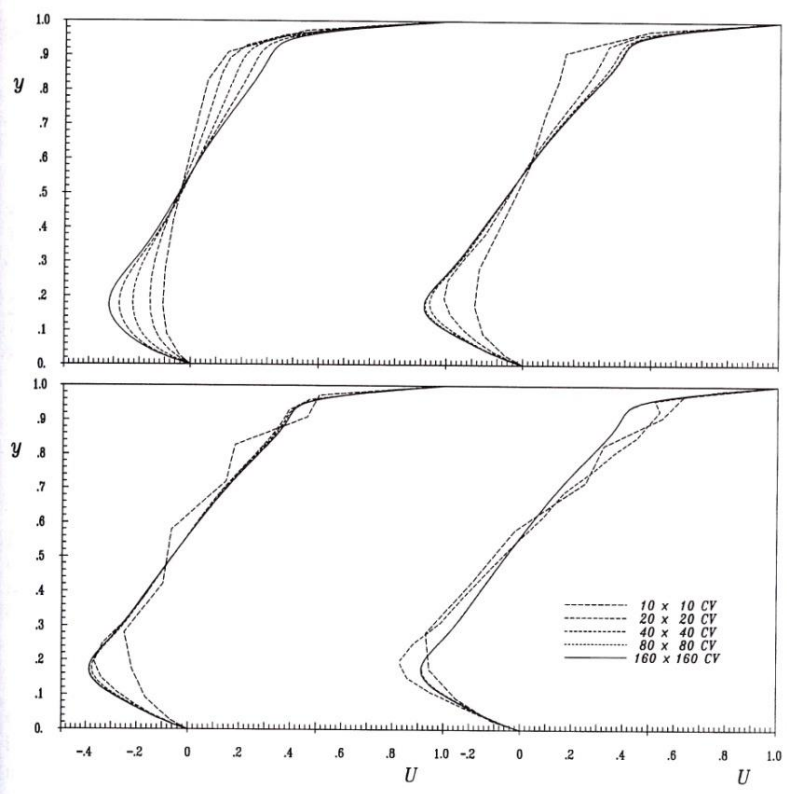
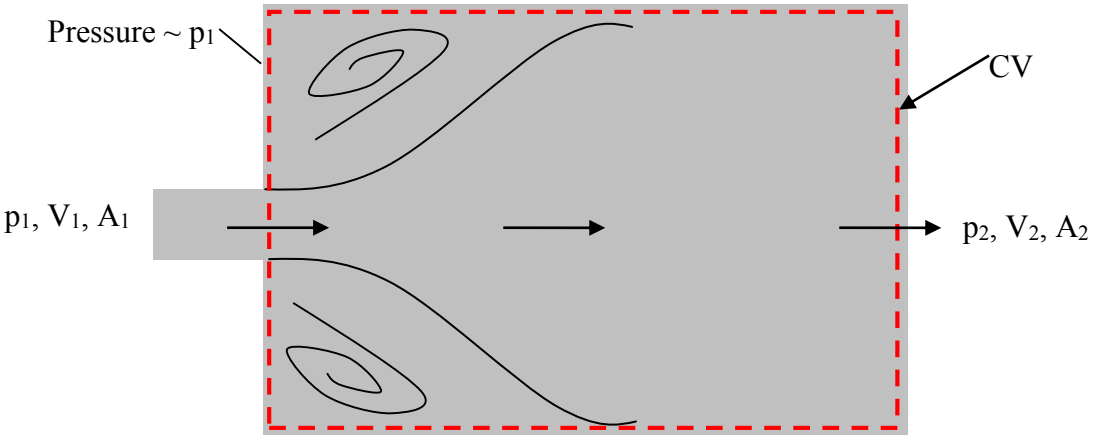


Fig. 7.12. Velocity profiles at the vertical centerline of the lid-driven cavity at Re = 1000, calculated on various grids using: midpoint rule and UDS (upper left), CDS (upper right) and cubic polynomial (lower left); Simpson's rule and cubic polynomials (lower right)

d) Recall the suddenly expanding pipe problem from Problem Set 2 (see below). Calculate the theoretical pressure difference for $A_1 = 1/3$, $A_2=1$, $V_1=1$. Using our Navier-Stokes solver, we will now compute this numerically. To run this test case, use TestCase(7,app), with:



app.nu=0.001 and app.nu=0.005, where app.nu sets the value of viscosity. For each case:
i) Calculate the numerical pressure change from the suddenly expanding entrance, to the end of the recirculation zone. (You may have to zoom in plots to see where the u-velocity reverses direction).
ii) Compare with the analytical value and briefly comment. In particular, how do results compare to Bernoulli's estimate? Is the pressure at the walls behind eddies close to p_1?
iii) How do the eddies vary with the Reynolds number? How does p_2 vary with the distance x along the pipe for the two different Reynolds numbers?

e) (Bonus: do this only if you are interested, you will get full credit without doing this part). Building on part b), run "TestCase(6)" with default app and "TestCase(6,app)" with "app.NonInc=1" and compare the final figure. A difference plot of the two cases may be useful. What is the difference between the two solvers? What accounts for the difference between the two solutions? Are the differences large? Why?

2.29 Numerical Fluid Mechanics

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