2.71/2.710 Optics   **Spring '13**
Practice Problems for the Final Exam   Posted Monday, May 13, 2013

1. **(Pedrotti 13-21)** A glass plate is sprayed with uniform opaque particles. When a distant point source of light is observed looking through the plate, a diffuse halo is seen whose angular width is about 2°. Estimate the size of the particles. (Hint: consider Fraunhoffer diffraction through random gratings, and use Babinet's principle)

Answer:
The diffraction pattern of an opaque circular particle is complementary to that due to circular apertures of the same size in an otherwise opaque screen.

Under the Fraunhofer condition ($\dfrac{k\left(x'^2+y'^2\right)}{2z} \ll 1, \quad \dfrac{k(x^2+y^2)}{2z} \ll 1$)

$$E(x',y') \approx \frac{1}{z} \iint \exp\left(-ik(\theta_{x'}x + \theta_{y'}y)\right)t(x,y)E(x,y)dxdy$$
$$\text{Where } \theta_{x'} \approx \frac{x'}{z}, \theta_{y'} \approx \frac{y'}{z}$$

For the given problem, we may further assume E(x, y) is a plane wave at normal incidence, and the transmission function t(x, y) for a single particle can be expressed as:

$$t(x,y) = 1 - circ(\frac{\sqrt{x^2+y^2}}{R})$$

Where R is the radius of the opaque particles.

$$E(x',y') \approx \frac{1}{z} \iint \exp\left(-ik(\theta_{x'}x + \theta_{y'}y)\right)\left[1 - circ(\frac{\sqrt{x^2+y^2}}{R})\right]dxdy$$
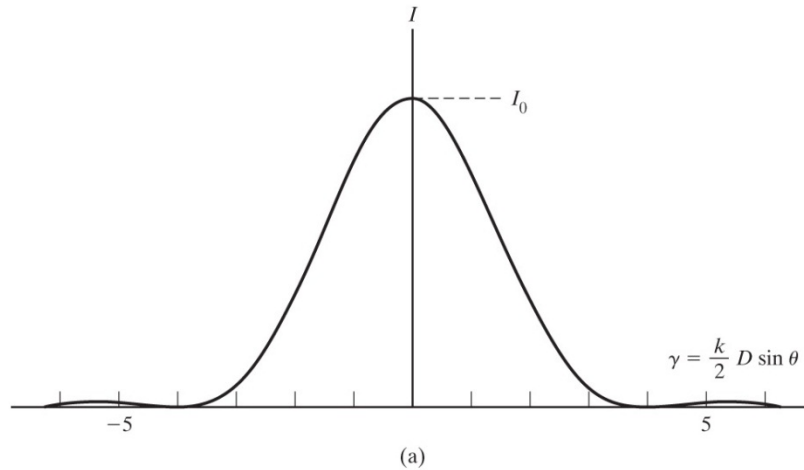
$$E(x',y') \approx \frac{1}{z}\mathcal{F}\left[1 - circ\left(\frac{\sqrt{x^2+y^2}}{R}\right)\right]$$

$$\text{With } x' = \frac{z}{k}k_x , y' = \frac{z}{k}k_y$$

$$E(k_x, k_y) \approx \frac{1}{z}\left[\delta(\sqrt{k_x{}^2 + k_y{}^2}) - |R|^2\frac{2\pi J_1\left(R\sqrt{k_x{}^2 + k_y{}^2}\right)}{R\sqrt{k_x{}^2 + k_y{}^2}}\right]$$

The halo is similar to an Airy disc!

We can evaluate the width of the halo (a second peak) based on the table on Figure 11_08 provided by Pedrotti:

(a)

	γ	$I/I_0 = (2J_1(\gamma)/\gamma)^2$
1st Maximum	0	1
1st Zero	3.832	0
2nd Maximum	5.136	0.0175
2nd Zero	7.016	0
3rd Maximum	8.417	0.00416
3rd Zero	10.173	0
4th Maximum	11.620	0.00160
4th Zero	13.324	0

(b)

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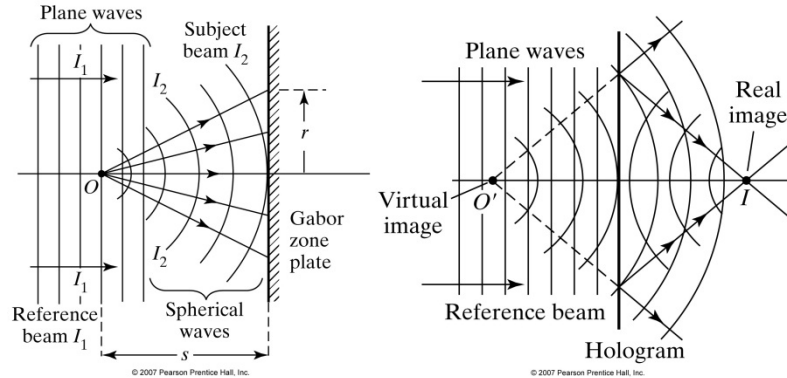Where $\gamma = R\sqrt{k_x^2 + k_y^2} = \frac{2\pi}{\lambda} R\theta.$

From the above table,

$$\frac{2\pi}{\lambda} R\Delta\theta = 7.106 - 3.832 = 3.274$$

Taking central wavelength at visible frequency, $\lambda = 500$ nm and given $\Delta\theta = 2^o$, we find the radius of the particle:

$$R = \lambda\frac{3.274}{(2\pi)^2 (\frac{\Delta\theta}{360})} = 500nm \times \left(\frac{3.274}{(2\pi)^2 \frac{2}{360}}\right) = 7463nm = 7.4\mu m$$

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2. (Adapted from Pedrotti 16-1 and 16-12)

Figure A. Recording (Left) and Reconstruction (Right) of a Gabor Hologram

a) Use the superposition of two beams to show that the recorded intensity pattern on a Gabor zone-plate (the hologram of a point source) is given approximately by
$$I = A + Bcos^2(ar^2)$$
Where $A = I_1 + I_2 - 2\sqrt{I_1 I_2}$, $B = 4\sqrt{I_1 I_2}$, and $a = \pi/(2s\lambda)$. Here I_1 and I_2 are the intensity due to the reference and signal beams, respectively, s is the distance of the object point from the film, and λ is the wavelength of the light. For the approximation, assume the path difference between the two beams is much smaller than s, so we are looking at the inner zones of the hologram.

Solution: in this problem, two beams are interfering at the zone plate: a reference plane wave with intensity I_1, and a spherical wave with intensity I_2. At a distance r from the symmetric axis, the path difference of the two beams can be written as:
$$\delta = \sqrt{s^2 + r^2} - s \approx \frac{r^2}{2s}$$
Therefore the intensity of the interference pattern can be written as:
$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}cos(k\delta)$$
And
$$cos(k\delta) = 2cos^2\left(\frac{k\delta}{2}\right) - 1$$
So we can rewrite the intensity into the following form:
$$I = \left[I_1 + I_2 - 2\sqrt{I_1 I_2}\right] + 4\sqrt{I_1 I_2}cos^2\left(\frac{k\delta}{2}\right)$$

$$\frac{k\delta}{2} = \frac{\pi r^2}{2\lambda s}$$

b) (2.710 only) Show that the phase delay of the diverging subject beam, at a point on the film at distance r from the axis, is given by $\pi r^2/ls$. This results follows when r<<s. Show also that the amplitude of the light transmitted by the film under illumination of the reference beam produces converging spherical wavefront, thus a real image on reconstruction.

Answer:

The path difference δ of the diverging beam with respect to the plane wave is derived in part (a). Therefore the phase delay is:

$$k\delta = \frac{\pi r^2}{\lambda s}$$

To answer last part of the question we can calculate the Fresnel diffraction pattern of this system using $k_x = k\frac{x'}{z}, k_y = k\frac{y'}{z}$.

$$E(x',y') \approx \iint \exp\left(ik\frac{x^2+y^2}{2z}\right)\left\{1 + \cos\left[k\frac{x^2+y^2}{2s}\right]\right\}\exp\{-i[k_x x + k_y y]\}dxdy$$

$$E(x',y') \approx \mathcal{F}\left\{\exp\left(ik\frac{x^2+y^2}{2z}\right) + \frac{1}{2}\exp\left[ik(x^2+y^2)\left(\frac{1}{2s}+\frac{1}{2z}\right)\right]\right.$$
$$\left. + \frac{1}{2}\exp\left[-ik(x^2+y^2)\left(\frac{1}{2s}-\frac{1}{2z}\right)\right]\right\}$$

The Fourier transform of the first term is straight forward:
$$\exp\left(-ik\frac{x'^2+y'^2}{2z}\right).$$

Likewise, we can express the second and the third term:
$$\frac{1}{2}\exp\left(-ik\frac{s}{2z}\frac{x'^2+y'^2}{s+z}\right) + \frac{1}{2}\exp\left(ik\frac{s}{2z}\frac{x'^2+y'^2}{(z-s)}\right).$$
the 3rd term indicates a converging wave front towards z=s (**a real image**) on the optical axis.

3. As shown in **Figure B**, the input images $t_1(x,y)$ and a phase mask $t_2(x',y')$ were placed on the surface of the lens L_1 and L_2. The focal length of the lens L1 is $f_1 = 2a$ and that of L_2 is $f_2 = a$. The spacing between L_1 and L_2 and the screen are all $2a$.
   a. Derive an expression for the light distribution on the screen.
   b. (**2.710 only**) Can you suggest a possible application of such an arrangement?
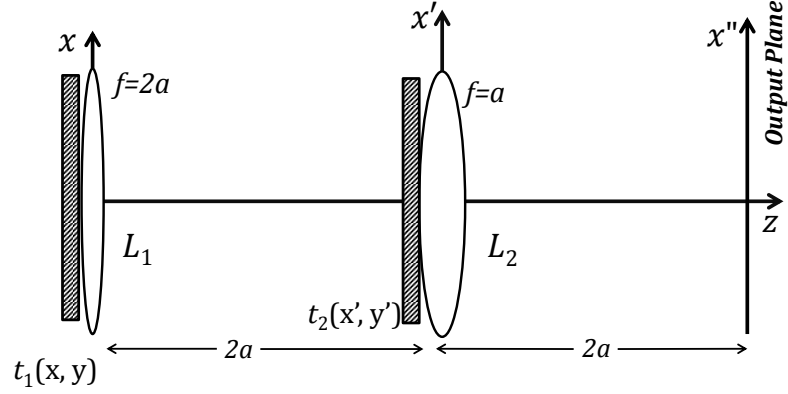

Figure B. Optical information processing using 2 convex lenses

<u>Answer:</u>
   a) This problem involves two steps of Fresnel- diffraction that are cascaded. If the illumination is a plane wave, the field behind the first lens can be written as:

$$E_+(x,y) = t_1(x,y)\exp\left[-ik\frac{(x^2+y^2)}{4a}\right]$$

We the let it propagate (z=2a) to the front surface of the second lens:

$E_-(x',y')$
$$= \frac{\exp(ik4a)}{4a}\exp(ik\frac{x'^2+y'^2}{4a})\iint \exp(-ik\frac{xx'+yy'}{2a})\exp(ik\frac{x^2+y^2}{4a})t_1(x,y)\exp\left[-ik\frac{(x^2+y^2)}{4a}\right]dxdy$$

$$E_-(x',y') = \frac{\exp(ik4a)}{4a}\exp(ik\frac{x'^2+y'^2}{4a})\iint \exp(-ik\frac{xx'+yy'}{2a})t_1(x,y)dxdy$$

$$E_-(x',y') = \frac{\exp(ik4a)}{4a}\exp(ik\frac{x'^2+y'^2}{4a})\boxed{\mathcal{F}(t_1(x,y))}$$

Then we can repeat the above step to the second lens (f=a):

$$E_+(x',y') = t_2(x',y')\exp\left[-ik\frac{(x'^2+y'^2)}{2a}\right]E_-(x',y')$$

Finally at the detector screen:

$E(x'',y'')$
$$= \frac{\exp(ik4a)}{4a}\exp(ik\frac{x''^2+y''^2}{4a})\iint \exp(-ik\frac{x''x'+y''y'}{2a})\exp(-ik\frac{x'^2+y'^2}{4a})t_2(x',y')E_-(x',y')dx'dy'$$

4. A simplified spectrometer setup is illustrated as in **Figure C**. The spectrometer consists of a sinusoidal amplitude grating with complex amplitude transmission function:

$$t(x) = t_0 + t_1 sin\left(\frac{2\pi}{\Lambda}x\right)$$

The grating is placed at the plane $z = 0$ and illuminated by an off–axis plane wave

$$E(x, z = 0) = E_0 exp(ikxsin\theta)$$

propagating at angle θ with respect to the optical axis z.
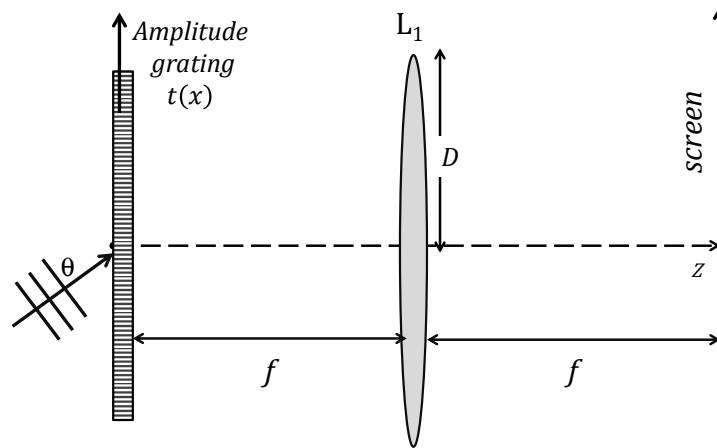

Figure C. A grating spectrometer.

The diffracted light waves are transmitted through a lens L_1 with focal length f, and collected at the back focal plane for wavelength analysis. In all problems below, let's assume the grating is sufficiently large and paraxial approximation is valid across the whole system.

A) With a **coherent** monochromatic illumination at wavelength λ, derive an expression for the Fresnel diffraction pattern right behind the grating, and argue that the expression that you derived physically represents a coherent superposition of three plane waves of different orientation.

<u>Answer:</u>

- **T**he field directly behind the grating is simply the product of the illumination field and the transmission function of the grating because spherical waves emanating from each point on the backside of the object has not propagated over any space yet.

$$E_+(x, y; z = 0) = E_{illum}(x, y)t(x, y)$$
$$= E_0 \exp\left(\frac{i2\pi}{\lambda}xsin\theta\right)\left(t_0 + t_1 \sin\left(\frac{2\pi}{\Lambda}x\right)\right) \quad (1)$$

We can rewrite the sine in the transmission function using exponentials:
$$\sin(ax) = \frac{1}{2i}e^{iax} - \frac{1}{2i}e^{-iax}$$
Equation (1) then becomes:
$$E_+(x,y;z=0) = E_0\left(t_0 e^{\frac{i2\pi}{\lambda}x\sin\theta} + \frac{t_1}{2i}\left(e^{i2\pi x\left(\frac{\sin\theta}{\lambda}+\frac{1}{\Lambda}\right)} - e^{i2\pi x\left(\frac{\sin\theta}{\lambda}-\frac{1}{\Lambda}\right)}\right)\right) \qquad (2)$$
This informs us that the diffraction pattern consists of three beams with incident angles $\left\{\theta, \left(\theta+\frac{\lambda}{\Lambda}\right), \left(\theta-\frac{\lambda}{\Lambda}\right)\right\}$. $(\sin\theta \approx \theta)$
The intensity of the field behind the grating:
$$I_+ \propto E_+ E_+^* =$$
$$E_0^2\left(t_0 e^{\frac{i2\pi}{\lambda}x\sin\theta} + \frac{t_1}{2i}\left(e^{i2\pi x\left(\frac{\sin\theta}{\lambda}+\frac{1}{\Lambda}\right)} - e^{i2\pi x\left(\frac{\sin\theta}{\lambda}-\frac{1}{\Lambda}\right)}\right)\right)\left(t_0 e^{\frac{-i2\pi}{\lambda}x\sin\theta} - \right.$$
$$\left. \frac{t_1}{2i}\left(e^{-i2\pi x\left(\frac{\sin\theta}{\lambda}+\frac{1}{\Lambda}\right)} - e^{-i2\pi x\left(\frac{\sin\theta}{\lambda}-\frac{1}{\Lambda}\right)}\right)\right) \qquad (3)$$

B) Using **geometric** optics, determine the location of the resulting spots on the back focal plane, neglecting for now the diffraction effect due to finite width D of the lens. Use your result to show that the wavelength detection limit, $\Delta\lambda$, of this spectrometer is inversely proportional to grating period Λ.

Answer:

- Using matrix method, we obtain:
$$\begin{pmatrix} x_{out} \\ \alpha_{out} \end{pmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x_{in} \\ \alpha_{in} \end{pmatrix} = \begin{bmatrix} 0 & f \\ -1/f & 0 \end{bmatrix}\begin{pmatrix} x_{in} \\ \alpha_{in} \end{pmatrix}$$
Therefore, $x_{out} = f\alpha_{in}$ for $\alpha_{in} = \left\{\theta, \left(\theta+\frac{\lambda}{\Lambda}\right), \left(\theta-\frac{\lambda}{\Lambda}\right)\right\}$,
$$x_{out} = \left\{f\theta, f\left(\theta+\frac{\lambda}{\Lambda}\right), f\left(\theta-\frac{\lambda}{\Lambda}\right)\right\}.$$
$$\frac{dx}{d\lambda} = \pm\frac{f}{\Lambda} \text{ or } d\lambda = \pm\frac{\Lambda}{f}dx$$
Where dx is approximately the pixel size of the CCD camera placed on the screen.

C) (**2.710 only**) Now taking into account of the diffraction effect due to finite width D of the lens, estimate the resulting spot width of the diffracted order at the back focal plane at monochromatic wavelength λ. Evaluate again the minimum spectral resolution $\Delta\lambda$ of the grating spectrometer.

Answer:

Under 1D approximation, the diffracted spot at back focal plane is approximately:

$$I_{\pm} \propto t_1^2 sinc^2 \left\{ x \left[\frac{D}{\lambda f} \right] \right\}$$

$$\text{Or } \delta x > \lambda \frac{f}{D}$$

Combined with part b), we obtain: $d\lambda = \pm \frac{\Lambda}{f} dx > \pm \lambda \frac{\Lambda}{D}$

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