22.101 Applied Nuclear Physics (Fall 2006)
Lecture 6 (9/27/06)

The Neutron-Proton System: Bound State of the Deuteron

References –
W. E. Meyerhof, *Elements of Nuclear Physics* (McGraw-Hill, New York, 1967), App. A.
B. L. Cohen, *Concepts of Nuclear Physics* (McGraw-Hill, New. York, 1971), chap. 3.

"The investigation of this nuclear force has turned out to be a truly monumental task: Perhaps more man-hours of work have been devoted to it than any other scientific question in the history of mankind" – B. L. Cohen, *Concepts of Nuclear Physics*, p. 32.

A direct way to gain some appreciation of the essential nature of nuclear force for our purposes is to consider the simplest possible nucleus, the two-nucleon system of the deuteron H^2 nucleus which is composed of a neturon and a proton. We will discuss two properties of this system using elementary quantum mechanics, the bound state of the nucleus (this lecture) and the scattering of a neutron by a proton (next lecture). The former problem is an application of our study of bound states in three-dimensions, while the latter is a new application of solving the time-independent Schrödinger wave equation. From these two problems a number of fundamental features of the nuclear potential will emerge.

Bound State of the Deuteron

The deuteron is the only stable bound system of two nucleons – neither the di-neutron nor the di-proton are stable. Experimentally it is known that the deuteron exists in a bound state of energy 2.23 Mev. This energy is the energy of a gamma ray given off in the reaction where a thermal neutron is absorbed by a hydrogen nucleus,

$$n + H^1 \rightarrow H^2 + \gamma \ (2.23 \text{ Mev})$$
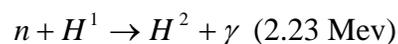
The inverse reaction of using electrons of known energy to produce external *bremsstrahlung* for (γ, n) reaction on H^2 also has been studied. Besides the ground state no stable excited states of H^2 have been found; however, there is a virtual state at ~ 2.30 Mev.

Suppose we combine the information on the bound-state energy of the deuteron with the bound-state solutions to the wave equation to see what we can learn about the potential well for neutron-proton interaction. Again we will assume the interaction between the two nucleons is of the form of a spherical well,

$$V(r) = -V_o \qquad\qquad r < r_o$$

$$= 0 \qquad\qquad r > r_o$$

We ask what is the energy level structure and what values should V_o and r_o take in order to be consistent with a bound state at energy $E_B = 2.23$ Mev.

There is good reason to believe that the deuteron ground state is primarily 1s (n=1, ℓ =0). First, the lowest energy state in practically all the model potentials is an s-state. Secondly, the magnetic moment of H^2 is approximately the sum of the proton and the neturon moments, indicating that $\underline{s}_n$ and $\underline{s}_p$ are parallel and no orbital motion of the proton relative to the neutron. This is also consistent with the total angular momentum of the ground state being I = 1.

We therefore proceed by considering only the $\ell = 0$ radial wave equation,

$$-\frac{\hbar}{m}\frac{d^2 u(r)}{dr^2} + V(r)u(r) = Eu(r) \qquad\qquad (6.1)$$

where E = - E_B, and m is the neutron (or proton) mass. In view of our previous discussions of bound-state calculations we can readily write down the interior and exterior solutions,

$$u(r) = A \sin Kr \qquad K = [m(V_o - E_B)]^{1/2} / \hbar \qquad r < r_o \qquad (6.2)$$

$$u(r) = Be^{-\kappa r} \qquad \kappa = \sqrt{mE_B} / \hbar \qquad r > r_o \qquad (6.3)$$

Applying the boundary condition at the interface, we obtain the relation between the potential well parameters, depth and width, and the bound-state energy E_B,

$$K \cot Kr_o = -\kappa, \quad \text{or} \quad \tan Kr_o = -\left(\frac{V_o - E_B}{E_B} \right)^{1/2} \qquad (6.4)$$

To go further we can consider either numerical or graphical solutions as discussed before, or approximations based on some special properties of the nuclear potential. Let us consider the latter option.

Suppose we take the potential well to be deep, that is, $V_o \gg E_B$, then the RHS (right hand side) of (6.4) is large and we get an approximate value for the argument of the tangent,

$$Kr_o \sim \pi / 2 \qquad (6.5)$$

Then,

$$K \sim \sqrt{mV_o} / \hbar \sim \pi / 2r_o$$

or

$$V_o r_o^2 \sim \left(\frac{\pi}{2} \right)^2 \frac{\hbar^2}{m} \sim 1 \text{ Mev-barn} \qquad (6.6)$$

We see that a knowledge of E_B allows us to determine the product of $V_o r_o^2$, and not V_o and r_o separately. From a study of neutron scattering by a proton which we will discuss in the next chapter, we will find that $r_o \sim 2F$. Putting this into (6.6) we get $V_o \sim 36$ Mev,

which justifies our taking V_o to be large compared to E_B. A sketch of the nuclear potential deduced from our calculation is shown in Fig. 1.
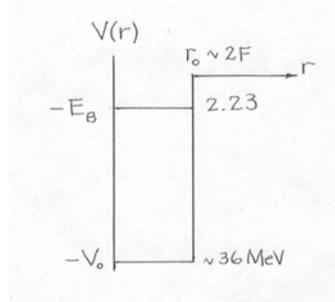
Fig. 1. The nuclear potential for the neutron-proton system in the form of a spherical well of depth V_o and width r_o. E_B is the bound-state energy of the deuteron.

We notice that since the interior wave function, sinKr, must match the exterior wave function, $\exp(-\kappa r)$, at the interface, the quantity Kr_o must be slightly greater than $\pi/2$ (a more accurate estimate gives 116° instead of 90°). If we write

$$K = 2\pi/\lambda \sim \pi/2r_o$$

then the 'effective wavelength' λ is approximately 4F, which suggests that much of the wave function is not in the interior region. The relaxation constant, or decay length, in the interior region can be estimated as

$$\frac{1}{\kappa} = \frac{\hbar}{\sqrt{mE_B}} \sim 4.3 \text{ F}$$

This means that the two nucleons in H^2 spend a large fraction of their time at $r > r_o$, the region of negative kinetic energy that is classically forbidden. We can calculate the root-mean-square radius of the deuteron wave function,

$$R_{rms}^2 = \frac{\int d^3r\, r^2 R^2(r)}{\int d^3r\, R^2(r)} = \frac{\int_0^\infty r^2 dr\, r^2 R^2(r)}{\int_0^\infty r^2 dr\, R^2(r)} \tag{6.7}$$

If we take $R(r) \sim e^{-\kappa r}$ for all r (this should result in an overestimate), we would obtain

$$R_{rms}^2 = \frac{\hbar}{\sqrt{2mE_B}} = 3F$$

This value can be compared with the estimate of nuclear radius based on the mass number,

$$(\text{radius})^2 \sim (1.4 \text{x} A^{1/3})^2 \sim 3.1 \text{ F}, \quad \text{or} \quad (1.2 \text{x} A^{1/3})^2 \sim 2.3 \text{ F}$$